PROBABILISTIC MODELING OF SELF EXCITING COUNTING PROCESSES WITH LATENT RISK RELAXATION

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Abstract. We study a class of self exciting counting processes with latent risk relaxation. Unlike a classical Poisson process, our model allows each event to temporarily increase the chance of future events. We couple the event count with a hidden risk variable that increases at event times and decreases between events. We derive the evolution equation for the joint law, obtain closed equations for the first order moments, and derive formulas for event times and inter event times. The model also provides a link between Poisson type dynamics, accumulation based self excitation, and exponential memory effects.

Keywords. self exciting counting processes; latent risk relaxation; joint law; event times; inter event times.

1 INTRODUCTION

Counting processes provide a fundamental framework for describing the random occurrence of events over time. In the classical Poisson setting, the event intensity is determined solely by an exogenous background rate [6]. In many practical situations, however, past events may temporarily increase the likelihood of future ones, so that a purely background driven model is no longer sufficient.

To account for such dependence, self-exciting counting models have been developed in several directions, most notably Hawkes type processes and related models with exponentially decaying memory [2, 5, 9, 10, 14, 16, 19]. Closely related Markovian and shot-noise based formulations have also been studied [4, 7, 8, 12, 18]. These models offer a more realistic description of event clustering than the Poisson framework, especially when the influence of previous events persists only for a limited period of time.

In this paper, we study a self-exciting counting process with latent risk relaxation. The model couples the counting process N_t with a latent nonnegative risk variable R_t, where R_t increases at event times and decays between successive events. The event intensity is taken in the form

$$\lambda^*(t) = \lambda(t) + \alpha R_t,$$

while the latent risk evolves according to

$$dR_t = -\beta R_t dt + \gamma dN_t.$$

Thus, each event increases the short-term risk of future events, while the resulting excitation gradually weakens in the absence of further jumps.

The motivation for this formulation is that, in many event driven systems, the effect of previous events is neither negligible nor permanent. Instead, one often observes a transient amplification mechanism: an event raises the short-term risk of subsequent events, and this elevated risk decays over time if no new event occurs. Such a structure is natural in applications involving clustered event arrivals, including traffic accident modeling and related event based systems [1, 3, 13, 15, 17].

The main contribution of the paper is a joint law analysis of the coupled process (N_t, R_t). This view-point leads to an evolution equation for the joint distribution, yields its basic probabilistic properties, and provides closed equations for the first-order moments. It also allows us to derive representation formulas for event times and inter-event times. In addition, the model clarifies how accumulation-type self-excitation and exponentially decaying memory can be represented within a unified Markovian framework [4, 7, 11].

The proposed formulation has two appealing features. First, it has a clear probabilistic interpretation, since it distinguishes the cumulative number of events from the current residual risk generated by the past. Second, it remains analytically tractable because the introduction of the latent risk variable preserves a Markovian structure. This makes it possible to combine a transparent modeling mechanism with a manageable distributional analysis.

The paper is organized as follows. Section 2 introduces the model and studies the joint law and moment equations. Section 3 is devoted to event times and inter-event times. Section 4 concludes the paper.

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2 COUPLED COUNT AND RISK MODEL AND DISTRIBUTIONAL DYNAMICS

In this section, we introduce the coupled count and risk model and derive the evolution equation satisfied by its joint law. We then establish its basic probabilistic properties and study the first order moments, with particular emphasis on the balance between excitation and relaxation.

We consider a stochastic process $\{N_t, t \geq 0\}$ taking values in $\mathbb{N}$, where N_t denotes the cumulative number of events that have occurred up to time t. To describe self excitation with relaxation, we couple N_t with another process $\{R_t, t \geq 0\}$, representing a latent non negative risk level that affects the occurrence of new events.

More precisely, for a small time step $0 < \Delta t \ll 1$, we assume that the count process evolves according to

$$N_{t+\Delta t} = N_t + H_t^{\Delta t}, \tag{2.1}$$

where $H_t^{\Delta t} \in \mathbb{N}$ is a random variable counting the number of events occurring in the interval $(t, t + \Delta t]$. We assume that $H_t^{\Delta t}$ satisfies

$$\text{Prob}\big(H_t^{\Delta t} = h\big) = \begin{cases} 1 - \lambda^*(t)\Delta t + o(\Delta t), & \text{if } h = 0, \\ \lambda^*(t)\Delta t + o(\Delta t), & \text{if } h = 1, \\ o(\Delta t), & \text{if } h \geq 2, \end{cases} \qquad h \in \mathbb{N}, \tag{2.2}$$

where $\lambda^*: [0, +\infty) \to \mathbb{R}_+$ denotes the event intensity. In contrast to background driven counting processes, we assume that λ^* depends on a latent risk variable R_t through

$$\lambda^*(t) = \lambda(t) + \alpha R_t, \tag{2.3}$$

where $\lambda: [0, +\infty) \to \mathbb{R}_+$ is a prescribed background intensity and $\alpha > 0$ is a coupling parameter. The latent risk evolves according to the relaxation jump dynamics

$$dR_t = -\beta R_t\, dt + \gamma\, dN_t, \qquad t > 0, \tag{2.4}$$

where $\beta > 0$ is the relaxation rate and $\gamma > 0$ is the jump size by which each event instantaneously increases the risk. The initial conditions are

$$N_0 = 0, \qquad R_0 = r_0 \geq 0. \tag{2.5}$$

The process R_t admits an explicit representation, which clarifies the connection between the present model and self exciting dynamics with exponential memory.

Proposition 2.1. *The latent risk process admits the representation*

$$R_t = r_0 e^{-\beta t} + \gamma \int_0^t e^{-\beta(t-s)}\, dN_s. \tag{2.6}$$

In particular, if $r_0 \geq 0$, then $R_t \geq 0$ for all $t \geq 0$, and hence

$$\lambda^*(t) \geq 0 \qquad \textit{for all } t \geq 0. \tag{2.7}$$

Proof. Equation (2.4) is linear in R_t. Multiplying by the integrating factor $e^{\beta t}$, we obtain

$$d\big(e^{\beta t} R_t\big) = \gamma e^{\beta t}\, dN_t.$$

Integrating over $[0, t]$ and using (2.5) we obtain (2.6). Since $r_0 \geq 0$, the process N_t is nondecreasing, and the kernel $e^{-\beta(t-s)}$ is non negative, it follows that $R_t \geq 0$ for all $t \geq 0$. The non negativity of $\lambda^*(t)$ then follows from (2.3). □

Remark 2.2. Substituting (2.6) into (2.3), we obtain

$$\lambda^*(t) = \lambda(t) + \alpha r_0 e^{-\beta t} + \alpha\gamma \int_0^t e^{-\beta(t-s)}\, dN_s. \tag{2.8}$$

Hence, the current intensity depends on the whole past history of the count process through an exponential kernel. In this sense, the system (2.3)–(2.4) provides a Markovian representation of a self exciting mechanism with exponentially decaying memory.

Remark 2.3. When $\beta = 0$, the latent risk reduces formally to

$$R_t = r_0 + \gamma \int_0^t d\, N_s.$$

Since $N_0 = 0$, this becomes

$$R_t = r_0 + \gamma N_t,$$

and the intensity becomes

$$\lambda^*(t) = \lambda(t) + \alpha r_0 + \alpha\gamma N_t.$$

Thus, in the absence of relaxation, the model reduces to an accumulation based self exciting structure. Conversely, when $\gamma = 0$, one has $R_t = r_0 e^{-\beta t}$, and the process reduces to a counting process driven by the background intensity together with a transient deterministic contribution.

The natural object of study is the joint distribution of the pair (N_t, R_t). To this end, for each $n \in \mathbb{N}$, we introduce the measure $p_n(\cdot, t)$ on $[0, \infty)$, defined by

$$p_n(A, t) := \text{Prob}(N_t = n,\ R_t \in A), \qquad A \subset [0, \infty) \text{ Borel.} \tag{2.9}$$

Whenever it is absolutely continuous with respect to the Lebesgue measure, we write

$$p_n(A, t) = \int_A p_n(r, t)\, dr.$$

At the initial time, we have

$$p_n(\cdot, 0) = \begin{cases} \delta_{r_0} & \text{if } n = 0, \\ 0 & \text{if } n \geq 1, \end{cases} \tag{2.10}$$

where δ_{r_0} denotes the Dirac mass at r_0.

We now derive the evolution equation satisfied by the joint law. This is the main structural result of the section.

Theorem 2.4. *Let $\varphi: \mathbb{N} \times \mathbb{R}_+ \to \mathbb{R}$ be such that, for each $n \in \mathbb{N}$, the map $r \mapsto \varphi(n, r)$ belongs to $C_c^1(\mathbb{R}_+)$. Then the joint law of (N_t, R_t) satisfies the weak evolution identity*

$$\begin{aligned} \frac{d}{dt} \sum_{n=0}^{\infty} \int_0^{\infty} \varphi(n, r) p_n(r, t)\, dr = \sum_{n=0}^{\infty} \int_0^{\infty} [-\beta r\, \partial_r \varphi(n, r) \\ + (\lambda(t) + \alpha r)\big(\varphi(n+1, r+\gamma) - \varphi(n, r)\big) p_n(r, t)\, dr. \end{aligned} \tag{2.11}$$

Assuming, in addition, that the measures $p_n(\cdot, t)$ admit sufficiently regular densities with respect to the Lebesgue measure, the corresponding strong form is formally given by

$$\begin{aligned} \partial_t p_n(r, t) + \partial_r\big(-\beta r\, p_n(r, t)\big) \\ = -(\lambda(t) + \alpha r) p_n(r, t) + \big(\lambda(t) + \alpha(r - \gamma)\big) p_{n-1}(r - \gamma, t), \end{aligned} \tag{2.12}$$

for $n = 0,1,2, \ldots$, with the conventions

$$p_{-1}(r, t) = 0, \qquad p_n(r, t) = 0 \quad \textit{for } r < 0.$$

In particular, for $n = 0$,

$$\partial_t p_0(r, t) + \partial_r\big(-\beta r\, p_0(r, t)\big) = -(\lambda(t) + \alpha r) p_0(r, t). \tag{2.13}$$

Proof. Let $\Delta t > 0$ be small. Conditionally on the state (N_t, R_t), the increment $H_t^{\Delta t}$ satisfies

$$\mathbb{P}\big(H_t^{\Delta t} = 0 \mid N_t, R_t\big) = 1 - (\lambda(t) + \alpha R_t)\Delta t + o(\Delta t),$$
$$\mathbb{P}\big(H_t^{\Delta t} = 1 \mid N_t, R_t\big) = (\lambda(t) + \alpha R_t)\Delta t + o(\Delta t),$$

and

$$\mathbb{P}\big(H_t^{\Delta t} \geq 2 \mid N_t, R_t\big) = o(\Delta t).$$

Moreover, by (2.4), on the event $\{H_t^{\Delta t} = 0\}$ one has

$$R_{t+\Delta t} = R_t - \beta R_t\, \Delta t + o(\Delta t),$$

whereas on the event $\{H_t^{\Delta t} = 1\}$,

$$R_{t+\Delta t} = R_t - \beta R_t\, \Delta t + \gamma + o(\Delta t).$$

Therefore,

$$\begin{aligned} \mathbb{E}[\varphi(N_{t+\Delta t}, R_{t+\Delta t}) \mid N_t, R_t] = (1 - (\lambda(t) + \alpha R_t)\Delta t)\varphi(N_t, R_t - \beta R_t \Delta t) \\ + (\lambda(t) + \alpha R_t)\Delta t\, \varphi(N_t + 1, R_t - \beta R_t \Delta t + \gamma) + o(\Delta t). \end{aligned}$$

Using the first order expansion

$$\varphi(N_t, R_t - \beta R_t \Delta t) = \varphi(N_t, R_t) - \beta R_t \Delta t\, \partial_r \varphi(N_t, R_t) + o(\Delta t),$$

and

$$\varphi(N_t + 1, R_t - \beta R_t \Delta t + \gamma) = \varphi(N_t + 1, R_t + \gamma) + o(1),$$

we obtain

$$\begin{aligned} \mathbb{E}[\varphi(N_{t+\Delta t}, R_{t+\Delta t}) \mid N_t, R_t] = \varphi(N_t, R_t) \\ + \Delta t\big[-\beta R_t\, \partial_r \varphi(N_t, R_t) + (\lambda(t) + \alpha R_t)\big(\varphi(N_t + 1, R_t + \gamma) - \varphi(N_t, R_t)\big)\big] + o(\Delta t). \end{aligned}$$

Taking expectation on both sides yields

$$\begin{aligned} \mathbb{E}[\varphi(N_{t+\Delta t}, R_{t+\Delta t})] = \mathbb{E}[\varphi(N_t, R_t)] \\ + \Delta t\, \mathbb{E}\big[-\beta R_t\, \partial_r \varphi(N_t, R_t) + (\lambda(t) + \alpha R_t)\big(\varphi(N_t + 1, R_t + \gamma) - \varphi(N_t, R_t)\big)\big] + o(\Delta t). \end{aligned}$$

Subtracting $\mathbb{E}[\varphi(N_t, R_t)]$, dividing by Δt, and letting $\Delta t \to 0$, we arrive at

$$\frac{d}{dt}\mathbb{E}[\varphi(N_t, R_t)] = \mathbb{E}\big[-\beta R_t\, \partial_r \varphi(N_t, R_t) + (\lambda(t) + \alpha R_t)\big(\varphi(N_t + 1, R_t + \gamma) - \varphi(N_t, R_t)\big)\big].$$

Writing the expectation in terms of the joint law gives (2.11).

Assume now that $p_n(\cdot, t)$ admits a sufficiently regular density for each n. Rewriting (2.11), we have

$$\sum_{n=0}^{\infty}\int_0^{\infty} \varphi\,(n,r)\,\partial_t p_n(r,t)\,dr = \sum_{n=0}^{\infty}\int_0^{\infty} -\beta r\,\partial_r\varphi(n,r)\,p_n(r,t)\,dr$$
$$+\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda(t)+\alpha r)\,\varphi(n+1,r+\gamma)p_n(r,t)\,dr - \sum_{n=0}^{\infty}\int_0^{\infty}(\lambda(t)+\alpha r)\,\varphi(n,r)p_n(r,t)\,dr.$$

Integrating by parts in the transport term, and using the compact support of $\varphi(n,\cdot)$ together with the factor r, we obtain

$$\sum_{n=0}^{\infty}\int_0^{\infty} \varphi\,(n,r)\,\partial_t p_n(r,t)\,dr = \sum_{n=0}^{\infty}\int_0^{\infty} \varphi\,(n,r)\,\partial_r\big(\beta r\,p_n(r,t)\big)\,dr$$
$$+\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda(t)+\alpha r)\,\varphi(n+1,r+\gamma)p_n(r,t)\,dr - \sum_{n=0}^{\infty}\int_0^{\infty}(\lambda(t)+\alpha r)\,\varphi(n,r)p_n(r,t)\,dr.$$

Next, in the jump term we shift the index and change variables, using the convention $p_n(r,t) = 0$ for $r < 0$. This gives

$$\sum_{n=0}^{\infty}\int_0^{\infty}(\lambda(t)+\alpha r)\,\varphi(n+1,r+\gamma)p_n(r,t)\,dr$$
$$=\sum_{n=0}^{\infty}\int_0^{\infty}\big(\lambda(t)+\alpha(r-\gamma)\big)\,\varphi(n,r)\,p_{n-1}(r-\gamma,t)\,dr.$$

Hence,

$$\sum_{n=0}^{\infty}\int_0^{\infty} \varphi\,(n,r)[\partial_t p_n(r,t) + \partial_r\big(-\beta r\,p_n(r,t)\big) + (\lambda(t)+\alpha r)p_n(r,t)$$
$$-\big(\lambda(t)+\alpha(r-\gamma)\big)p_{n-1}(r-\gamma,t)\,dr = 0.$$

Since φ is arbitrary, this formally yields (2.12). The case $n = 0$ immediately gives (2.13). □

We next verify that the weak evolution identity is consistent with conservation of total probability.

Proposition 2.5. *Assuming (2.10), the total mass is conserved, namely*

$$\sum_{n=0}^{\infty}\int_0^{\infty} p_n\,(r,t)\,dr = 1 \qquad \textit{for all } t > 0. \tag{2.14}$$

Proof. Taking $\varphi \equiv 1$ in (2.11), we have $\partial_r\varphi = 0$ and

$$\varphi(n+1, r+\gamma) - \varphi(n,r) = 0.$$

Hence,

$$\frac{d}{dt}\sum_{n=0}^{\infty}\int_0^{\infty} p_n\,(r,t)\,dr = 0.$$

Therefore, the total mass is constant in time. Using the initial condition (2.10), we obtain

$$\sum_{n=0}^{\infty}\int_0^{\infty} p_n\,(r,0)\,dr = \int_0^{\infty}\delta_{r_0}\,(r)\,dr = 1.$$

This yields (2.14). □

Remark 2.6. The non negativity of $p_n(\cdot, t)$ follows from its probabilistic definition as the joint law of (N_t, R_t).

A direct consequence of the probabilistic interpretation of $p_n(\cdot, t)$ together with (2.14) is that the marginal law of the count variable,

$$f(n,t) := \int_0^\infty p_n(r,t)\,dr = \mathrm{Prob}(N_t = n), \tag{2.15}$$

is a probability distribution on $\mathbb{N}$ for every $t \geq 0$.

We now turn to the first order moments of the process. Define the mean event count and the mean latent risk by

$$m(t) := \mathbb{E}[N_t] = \sum_{n=0}^{\infty} \int_0^\infty n\; p_n(r,t)\,dr, \tag{2.16}$$

and

$$q(t) := \mathbb{E}[R_t] = \sum_{n=0}^{\infty} \int_0^\infty r\; p_n(r,t)\,dr. \tag{2.17}$$

Proposition 2.7. *The mean event count and the mean latent risk satisfy*

$$\frac{dm}{dt} = \lambda(t) + \alpha q(t), \tag{2.18}$$

$$\frac{dq}{dt} = -\beta q(t) + \gamma\big(\lambda(t) + \alpha q(t)\big), \tag{2.19}$$

with initial conditions

$$m(0) = 0, \qquad q(0) = r_0. \tag{2.20}$$

Proof. The identities below are obtained from (2.11) by using the test functions $\varphi(n,r) = n$ and $\varphi(n,r) = r$, justified by approximation with compactly supported functions.

Taking $\varphi(n,r) = n$, we have $\partial_r \varphi = 0$ and

$$\varphi(n+1, r+\gamma) - \varphi(n,r) = 1.$$

Hence

$$\frac{d}{dt} \sum_{n=0}^{\infty} \int_0^\infty n\; p_n(r,t)\,dr = \sum_{n=0}^{\infty} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\,dr.$$

Using (2.16), (2.17), and (2.14), we obtain

$$\frac{dm}{dt} = \lambda(t) + \alpha q(t).$$

Next, taking $\varphi(n,r) = r$, we have $\partial_r \varphi = 1$ and

$$\varphi(n+1, r+\gamma) - \varphi(n,r) = \gamma.$$

Therefore,

$$\frac{d}{dt} \sum_{n=0}^{\infty} \int_0^\infty r\; p_n(r,t)\,dr = \sum_{n=0}^{\infty} \int_0^\infty [-\beta r + \gamma(\lambda(t) + \alpha r)]\, p_n(r,t)\,dr.$$

Using (2.17) and (2.14), we obtain

$$\frac{dq}{dt} = -\beta q(t) + \gamma\big(\lambda(t) + \alpha q(t)\big).$$

Finally, (2.10) gives

$$m(0) = 0, \qquad q(0) = \int_0^\infty r\; \delta_{r_0}(dr) = r_0.$$

This proves (2.20). □

The previous proposition shows that the latent risk acts as an intermediate state linking the event count to the self exciting contribution. It also leads to a simple balance relation between excitation and relaxation at the level of the mean risk.

Theorem 2.8. *The mean latent risk satisfies*

$$\frac{dq}{dt} = (\alpha\gamma - \beta) q(t) + \gamma\lambda(t). \tag{2.21}$$

Consequently, the qualitative behavior of $q(t)$ is governed by the sign of the balance parameter

$$\alpha\gamma - \beta.$$

More precisely:

- *if $\alpha\gamma < \beta$, then the mean risk dynamics is relaxation dominated;*

- *if $\alpha\gamma = \beta$, then the system is in a critical balance regime;*
- *if $\alpha\gamma > \beta$, then the mean risk dynamics is excitation dominated.*

Moreover, if $\lambda \in L^1(\mathbb{R}_+)$ and $\alpha\gamma < \beta$, then $q(t)$ remains bounded for all $t \geq 0$.

Proof. Equation (2.21) follows directly from (2.19):

$$\frac{dq}{dt} = -\beta q(t) + \gamma\big(\lambda(t) + \alpha q(t)\big) = (\alpha\gamma - \beta)q(t) + \gamma\lambda(t).$$

Hence the behavior of the mean latent risk is determined by the sign of the coefficient

$$\alpha\gamma - \beta.$$

If $\alpha\gamma < \beta$, the contribution of q is damped by relaxation; if $\alpha\gamma = \beta$, the linear part is balanced; if $\alpha\gamma > \beta$, the self exciting contribution prevails over the relaxation term.

It remains to prove the boundedness of q when $\lambda \in L^1(R_+)$ and $\alpha\gamma < \beta$. Set

$$a := \alpha\gamma - \beta < 0.$$

Then (2.21) becomes

$$q'(t) = a\, q(t) + \gamma\lambda(t), \qquad q(0) = r_0.$$

By the variation of constants formula,

$$q(t) = e^{at} r_0 + \gamma \int_0^t e^{a(t-s)}\, \lambda(s)\, ds.$$

Since $a < 0$, we have $e^{a(t-s)} \leq 1$ for $0 \leq s \leq t$. Therefore,

$$q(t) \leq r_0 + \gamma \int_0^t \lambda\,(s)\, ds \leq r_0 + \gamma \int_0^\infty \lambda\,(s)\, ds.$$

Because $\lambda \in L^1(\mathbb{R}_+)$, the right hand side is finite and independent of t. Hence $q(t)$ remains bounded on $[0, \infty)$. □

Remark 2.9. Under the assumptions of Theorem 2.8, if $\alpha\gamma < \beta$ and $\lambda \in L^1(\mathbb{R}_+)$, then $q(t)$ remains bounded. Hence, by (2.18), the mean count satisfies

$$m'(t) = \lambda(t) + \alpha q(t) \leq \lambda(t) + C$$

for some constant $C > 0$. Thus, the mean count is driven by the background intensity together with a uniformly bounded self exciting correction. This contrasts with accumulation based models, where no relaxation mechanism limits the growth of the self exciting contribution.

For completeness, we also introduce the second order quantities

$$E_N(t) := \mathbb{E}[N_t^2], \qquad E_R(t) := \mathbb{E}[R_t^2], \qquad C(t) := \mathbb{E}[N_t R_t]. \tag{2.22}$$

Using (2.11), formally or by approximation with compactly supported test functions, one may derive a corresponding system for these moments by taking $\varphi(n, r) = n^2$, $\varphi(n, r) = r^2$, and $\varphi(n, r) = nr$. Since the resulting formulas are longer and not needed in what follows, we do not pursue them here.

A particularly instructive special case is obtained when the background intensity is constant:

$$\lambda(t) \equiv \lambda_0 > 0. \tag{2.23}$$

Then (2.18)–(2.19) become

$$\frac{dm}{dt} = \lambda_0 + \alpha q(t), \qquad \frac{dq}{dt} = (\alpha\gamma - \beta)q(t) + \gamma\lambda_0, \tag{2.24}$$

with (2.20). If $\alpha\gamma \neq \beta$, the explicit solution of the second equation is

$$q(t) = r_0 e^{(\alpha\gamma-\beta)t} + \gamma\lambda_0 \frac{e^{(\alpha\gamma-\beta)t} - 1}{\alpha\gamma - \beta}, \tag{2.25}$$

and therefore

$$m(t) = \lambda_0 t + \alpha \int_0^t q\,(s)\, ds. \tag{2.26}$$

If instead $\alpha\gamma = \beta$, then

$$q(t) = r_0 + \gamma\lambda_0 t, \tag{2.27}$$

whence

$$m(t) = \lambda_0 t + \alpha r_0 t + \frac{\alpha\gamma\lambda_0}{2} t^{2.} \tag{2.28}$$

These formulas clearly distinguish the three regimes: bounded mean risk and asymptotically linear growth when relaxation dominates, linear growth of the mean risk and quadratic growth of the mean count in the

critical case, and exponentially amplified behavior when excitation dominates. Section 3 is devoted to the analysis of event times and inter event times for the process introduced here.

3 EVENT TIMES AND INTER EVENT TIMES

In this section, we investigate the temporal structure of the process introduced in Section 2. More precisely, we study the law of the event times and of the inter event times. In contrast to simpler background driven counting models, the presence of the latent risk variable makes these quantities depend on the full joint count and risk distribution rather than on the count variable alone.

3.1 Law of the event times

We next investigate the distribution of the event times. For $k \geq 1$, let T_k be the occurrence time of the k-th event. Unlike the cumulative count, the law of T_k gives direct information on when successive events occur. In our model, this law depends on both the event count and the latent risk level. We derive it from the joint law of the coupled count and risk process.

For $k \in \mathbb{N}$, $k \geq 1$, we define the k-th event time by

$$T_k := \inf\{t \geq 0 : N_t = k\}. \tag{3.1}$$

By construction, the event $\{T_k \leq t\}$ is equivalent to $\{N_t \geq k\}$, and therefore

$$\text{Prob}(T_k \leq t) = \text{Prob}(N_t \geq k) = 1 - \sum_{n=0}^{k-1} \text{Prob}\,(N_t = n) = 1 - \sum_{n=0}^{k-1} f\,(n,t), \tag{3.2}$$

where, consistently with (2.15),

$$f(n,t) = \int_0^\infty p_n\,(r,t)\,dr.$$

Assuming that T_k admits a density g_k, differentiating (3.2) with respect to time gives

$$g_k(t) = -\sum_{n=0}^{k-1} \partial_t\, f(n,t) = -\sum_{n=0}^{k-1} \int_0^\infty \partial_t\, p_n(r,t)\,dr.$$

Using the formal strong form (2.12), and assuming sufficient regularity, we obtain

$$\partial_t p_n(r,t) = -\partial_r\big(-\beta r\, p_n(r,t)\big) - (\lambda(t) + \alpha r) p_n(r,t) + \big(\lambda(t) + \alpha(r-\gamma)\big) p_{n-1}(r-\gamma,t).$$

Hence

$$g_k(t) = \sum_{n=0}^{k-1} \int_0^\infty \partial_r\left(-\beta r\, p_n(r,t)\right) dr$$

$$+ \sum_{n=0}^{k-1} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\,dr - \sum_{n=0}^{k-1} \int_0^\infty \big(\lambda(t) + \alpha(r-\gamma)\big)\, p_{n-1}(r-\gamma,t)\,dr.$$

Assuming that the boundary term at infinity vanishes, the transport contribution is zero. After the change of variables $r = \rho + \gamma$ in the last integral and a shift of indices, we arrive at

$$g_k(t) = \sum_{n=0}^{k-1} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\,dr - \sum_{n=0}^{k-2} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\,dr,$$

which yields the following formula.

Theorem 3.1. *Assume that T_k admits a density g_k, and that the strong form* (2.12) *holds with sufficient regularity so that the following computations are justified. Then*

$$g_k(t) = \int_0^\infty (\lambda(t) + \alpha r)\, p_{k-1}(r,t)\,dr. \tag{3.3}$$

Proof. From (3.2),

$$\mathbb{P}(T_k \leq t) = 1 - \sum_{n=0}^{k-1} f\,(n,t),$$

hence, differentiating with respect to t,

$$g_k(t) = -\sum_{n=0}^{k-1} \partial_t f(n,t) = -\sum_{n=0}^{k-1} \int_0^\infty \partial_t p_n(r,t)\, dr.$$

Using the strong form (2.12), we obtain

$$\partial_t p_n(r,t) = -\partial_r\big(-\beta r\, p_n(r,t)\big) - (\lambda(t) + \alpha r) p_n(r,t) + \big(\lambda(t) + \alpha(r-\gamma)\big) p_{n-1}(r-\gamma,t).$$

Therefore,

$$g_k(t) = \sum_{n=0}^{k-1} \int_0^\infty \partial_r\left(-\beta r\, p_n(r,t)\right) dr + \sum_{n=0}^{k-1} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\, dr$$
$$- \sum_{n=0}^{k-1} \int_0^\infty \big(\lambda(t) + \alpha(r-\gamma)\big)\, p_{n-1}(r-\gamma,t)\, dr.$$

Assuming that the boundary term at infinity vanishes, the transport contribution is zero. In the last sum, after the change of variables $r = \rho + \gamma$ and a shift of index, using the convention $p_{-1} = 0$, we obtain

$$\sum_{n=0}^{k-1} \int_0^\infty \big(\lambda(t) + \alpha(r-\gamma)\big)\, p_{n-1}(r-\gamma,t)\, dr = \sum_{n=0}^{k-2} \int_0^\infty (\lambda(t) + \alpha\rho)\, p_n(\rho,t)\, d\rho.$$

Hence

$$g_k(t) = \sum_{n=0}^{k-1} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\, dr - \sum_{n=0}^{k-2} \int_0^\infty (\lambda(t) + \alpha r)\, p_n(r,t)\, dr.$$

Subtracting the second sum from the first leaves only the term corresponding to $n = k-1$. Therefore,

$$g_k(t) = \int_0^\infty (\lambda(t) + \alpha r)\, p_{k-1}(r,t)\, dr.$$

This proves (3.3). □

Formula (3.3) has a clear probabilistic interpretation: for the k-th event to occur at time t, the system must be in the state $N_t = k-1$ with latent risk level r, and an event must then occur with instantaneous intensity $\lambda(t) + \alpha r$.

A first explicit case is obtained when $k = 1$. Since initially $N_0 = 0$, (3.3) gives

$$g_1(t) = \int_0^\infty (\lambda(t) + \alpha r)\, p_0(r,t)\, dr. \tag{3.4}$$

Before the first event occurs, the risk evolves deterministically according to

$$R_t = r_0 e^{-\beta t},$$

so that, in the sense of measures,

$$p_0(r,t) = f(0,t)\, \delta_{r_0 e^{-\beta t}}(r).$$

Moreover, before the first jump the intensity is deterministic and equal to

$$\lambda(t) + \alpha r_0 e^{-\beta t}.$$

Hence the probability that no event has occurred up to time t is

$$f(0,t) = \exp\left(-\int_0^t \big(\lambda(s) + \alpha r_0 e^{-\beta s}\big)\, ds\right). \tag{3.5}$$

Substituting this into (3.4), we obtain

$$g_1(t) = \big(\lambda(t) + \alpha r_0 e^{-\beta t}\big) \exp\left(-\int_0^t \big(\lambda(s) + \alpha r_0 e^{-\beta s}\big)\, ds\right). \tag{3.6}$$

Remark 3.2. If $r_0 = 0$, then (3.6) reduces to

$$g_1(t) = \lambda(t) e^{-\Lambda(t)}, \qquad \Lambda(t) := \int_0^t \lambda(s)\, ds,$$

which coincides with the first event time density of a background driven counting process. This is expected, since when the initial risk vanishes there is no self exciting contribution before the first jump.

3.2 Law of the inter event times

We now study the law of the inter event times, which describes the waiting times between consecutive events. This quantity provides a more local view of the temporal structure of the counting process than the cumulative count alone. In particular, it helps explain how the occurrence of one event influences the timing of the next one. In the present model, such waiting times depend not only on the background intensity but also on the latent risk carried over from previous events. We therefore derive their distribution from the joint law of the coupled count and risk process.

We now turn to the inter event times

$$\Delta T_k := T_k - T_{k-1}, \qquad k \in \mathbb{N},\ k \geq 1, \tag{3.7}$$

where, by convention, $T_0 := 0$. In particular,

$$\Delta T_1 = T_1,$$

so that its density is simply

$$h_1(\tau) = g_1(\tau), \qquad \tau \geq 0. \tag{3.8}$$

For $k \geq 2$, the distribution of ΔT_k is more involved, since the event times are not independent and the latent risk introduces an additional memory effect. We therefore condition on both the previous event time and the post jump risk level.

Let $s \geq 0$and $\rho \geq 0$. Suppose that the $(k-1)$-st event occurs at time s, and that the latent risk immediately after that event equals ρ. Here $R_{T_{k-1}}$ is understood as the post jump risk. Between the $(k-1)$-st and k-th events, the risk evolves as

$$R_t = \rho e^{-\beta(t-s)}, \qquad t \geq s.$$

Accordingly, the conditional event intensity becomes

$$\lambda(t) + \alpha\rho e^{-\beta(t-s)}.$$

Therefore, the conditional survival probability of the waiting time to the next event is

$$\text{Prob}\big(T_k > t \mid T_{k-1} = s,\ R_{T_{k-1}} = \rho\big) = \exp\left(-\int_s^t \big(\lambda(u) + \alpha\rho e^{-\beta(u-s)}\big)\, du\right),$$

and the corresponding conditional density is

$$\begin{aligned} &g_{T_k \mid T_{k-1}, R_{T_{k-1}}}(t \mid s, \rho) \\ &\quad = \big(\lambda(t) + \alpha\rho e^{-\beta(t-s)}\big) \exp\left(-\int_s^t \big(\lambda(u) + \alpha\rho e^{-\beta(u-s)}\big)\, du\right), \qquad t > s. \end{aligned} \tag{3.9}$$

Assuming that the corresponding regular conditional law exists, let $\nu_{k-1}(d\rho \mid s)$ denote the conditional law of the post jump latent risk given $T_{k-1} = s$. Averaging (3.9) with respect to ν_{k-1} yields

$$\begin{aligned} g_{T_k \mid T_{k-1}}(t \mid s) = \int_0^\infty &\big(\lambda(t) \\ &+ \alpha\rho e^{-\beta(t-s)}\big) \exp\left(-\int_s^t \big(\lambda(u) + \alpha\rho e^{-\beta(u-s)}\big)\, du\right) \nu_{k-1}(d\rho \mid s), t > s. \end{aligned} \tag{3.10}$$

Accordingly, whenever the corresponding densities exist, the joint density of (T_{k-1}, T_k) is given by

$$g_{k-1,k}(s,t) = g_{T_k \mid T_{k-1}}(t \mid s)\, g_{k-1}(s), \qquad t > s. \tag{3.11}$$

The density of the inter event time ΔT_k is then recovered by integration:

$$h_k(\tau) = \int_0^\infty g_{k-1,k}\,(s, s+\tau)\, ds, \qquad \tau \geq 0. \tag{3.12}$$

Combining (3.10) and (3.11), we obtain the following representation.

Proposition 3.3. *Under the above assumptions, for every $k \in \mathbb{N}$, $k \geq 2$, the density of the k-th inter event time is given by*

$$\begin{aligned} h_k(\tau) = \int_0^\infty g_{k-1}\,(s) \Bigg[\int_0^\infty &\big(\lambda(s+\tau) + \alpha\rho e^{-\beta\tau}\big) \\ \times \exp\left(-\int_s^{s+\tau} \big(\lambda(u) + \alpha\rho e^{-\beta(u-s)}\big)\, du\right) &\nu_{k-1}(d\rho \mid s)\Bigg] ds. \end{aligned} \tag{3.13}$$

Proof. By definition,

$$\Delta T_k = T_k - T_{k-1}, \qquad k \geq 2.$$

Hence, for $\tau \geq 0$, its density is obtained from the joint density of (T_{k-1}, T_k) through

$$h_k(\tau) = \int_0^\infty g_{k-1,k}\,(s, s+\tau)\,ds. \tag{3.14}$$

Using (3.11) and (3.10), we find

$$g_{k-1,k}(s, s+\tau) = g_{k-1}(s) \int_0^\infty \left(\lambda(s+\tau) + \alpha\rho e^{-\beta\tau}\right) \times \exp\left(-\int_s^{s+\tau} \left(\lambda(u) + \alpha\rho e^{-\beta(u-s)}\right)\,du\right) \nu_{k-1}(d\rho \mid s).$$

Substituting this into (3.14) yields (3.13). □

Formula (3.13) shows that, in general, the inter event times are not identically distributed. Indeed, both the distribution of the previous event time and the conditional law of the post jump latent risk may depend on the event index, which makes the family $\{h_k\}_{k\geq 1}$ nonstationary.

Remark 3.4. The representation (3.13) simplifies considerably when the conditional law $\nu_{k-1}(d\rho \mid s)$ is concentrated around a deterministic value, or when it is replaced by a suitable approximation based on conditional expectations. This may be useful for numerical purposes and asymptotic approximations.

3.3 Aggregate inter event time

Assuming that $m(T) > 0$, it is also useful to introduce a single random variable representing a typical inter event time over a finite observation horizon $T > 0$. To this end, we define a random index $K \in \mathbb{N}$, $K \geq 1$, by

$$\text{Prob}(K = k) := \frac{\text{Prob}(T_k \leq T)}{m(T)}, \qquad k \geq 1,$$

where $m(T) = \mathbb{E}[N_T]$ is the mean number of events observed up to time T.

This definition is well posed, since

$$\sum_{k=1}^{\infty} \text{Prob}\,(T_k \leq T) = \sum_{k=1}^{\infty} \text{Prob}\,(N_T \geq k) = \mathbb{E}[N_T] = m(T).$$

We then define ΔT_K as the aggregate inter event time. Its density, denoted by $h = h(\tau)$, is given by

$$h(\tau) = \sum_{k=1}^{\infty} h_k\,(\tau)\,\text{Prob}(K = k) = \frac{1}{m(T)} \sum_{k=1}^{\infty} h_k\,(\tau)\,\text{Prob}(T_k \leq T), \qquad \tau \geq 0.$$

Thus, $h(\tau)$ provides a weighted average description of the waiting times between consecutive events over the observation window $[0, T]$.

4 CONCLUSIONS

We introduced a class of self exciting counting processes with latent risk relaxation, where the event count is coupled with a non negative hidden risk variable that increases at event times and decays between events. This framework links accumulation based self excitation with exponential memory dynamics.

We derived the evolution equation for the joint law, obtained a closed system for the first order moments, identified the balance parameter governing excitation and relaxation, and derived representation formulas for event times and inter event times.

These results show that latent risk relaxation provides a natural and tractable framework for self exciting dynamics with temporary memory effects. Possible extensions include higher order moments, statistical inference, and multivariate or spatial models.

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