THE COMMUTING GRAPHS OF SOLVABLE LIE ALGEBRAS HAVING ABELIAN DERIVED ALGEBRAS OF DIMENSION 1 OR CODIMENSION 1

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Abstract. In this paper, we give an explicit description of the commuting graph of a subclass of solvable Lie algebras over an arbitrary field of characteristic not equal to 2 and investigate its connectivity. In the case where the graph is connected, its diameter is determined; otherwise, when the graph is disconnected, all of its connected components are explicitly described.

1. Introduction

Lie algebra theory is a central area of modern mathematics, with far-reaching applications in geometry, theoretical physics, and various other fields. According to the Levi–Malcev Theorem, any finite-dimensional Lie algebra can be decomposed into a semidirect sum of a semisimple Lie algebra and a solvable one. Consequently, the classification and study of general Lie algebras are typically reduced to those of semisimple and solvable Lie algebras. Although semisimple Lie algebras have been completely classified since the pioneering works of Cartan (see [5]) and Gantmacher (see [8]), the classification and analysis of solvable Lie algebras remain largely open problems.

In recent years, one of the approaches that has attracted considerable attention from mathematicians to investigate the properties of algebraic structures is through graph theory. By associating a graph with an algebraic structure, one can leverage powerful tools from graph theory to explore algebraic properties in a more visual and intuitive manner. Along this line of research, extensive studies have been conducted on algebraic structures such as groups (see [1], [4]), rings (see [3], [7]) and algebras (see [2], [11]). However, the application of graph-theoretic methods to Lie algebras, particularly solvable ones, remains relatively recent and underdeveloped. To our knowledge, Wang and Xia (see [14]) introduced the first notion of a commuting graph associated with a Lie algebra in 2017 in their study of commuting graphs of simple Lie algebras. To date, this remains perhaps a rare work on this subject. In their formulation, the commuting graph of a Lie algebra is a simple undirected graph whose vertices consist of all noncentral elements, and two vertices are adjacent if and only if the corresponding elements commute (i.e., their Lie bracket is zero). Studying the commuting graph reveals the commutation relationships among elements, thereby shedding light on the internal structure of the Lie algebra.

This paper aims to investigate the commuting graphs of a subclass of solvable Lie algebras, offering a new perspective on this class of algebras. Specifically, we provide an explicit description of the commuting graphs, including their vertex sets, edge sets, and the cardinality of these sets in the finite case, for Lie algebras whose derived algebras are one-dimensional and one-codimensional. Furthermore, we determine the diameter of the commuting graph in the case where it is connected. In the disconnected case, we explicitly describe all its connected components. Through this analysis, we aim to offer a more comprehensive understanding of the structure of solvable Lie algebras from the viewpoint of graph theory.

2. Preliminaries

In this section, we briefly recall some fundamental concepts from graph theory and Lie algebra theory. For more detailed treatments, the reader is referred to [6] and [13].

An *undirected simple graph* (or simply a *graph*) is an ordered pair $G = (V, E)$, where $V \neq \emptyset$ is *vertex set* and

$$E \subseteq \{\{x, y\} \mid x, y \in V \text{ and } x \neq y\}$$

2020 *Mathematics Subject Classification.* 22E25; 22E27; 05C07; 05C40.

Key words and phrases. commuting graph, solvable Lie algebra, diameter, connectivity.

is *edge set*. Two distinct vertices u and v are said to be *adjacent* if $\{u, v\} \in E$. In this case, u and v are called the *endpoints* of the edge $\{u, v\}$. A graph is called *complete* if every pair of distinct vertices is adjacent. The complete graph on the n vertices is denoted by K_n, and the complete graph on a vertex subset $S \subseteq V$ is denoted by K_S. A *path* from a vertex u to a vertex v is a sequence of distinct vertices $\{v_1, v_2, \ldots, v_k\}$ such that $v_1 = u$, $v_k = v$, and $\{v_i, v_{i+1}\} \in E$ for all $i = 1, 2, \ldots, k-1$. The *length* of the path $\{v_1, v_2, \ldots, v_k\}$ is defined as $k-1$. The *distance* between two vertices u and v, denoted by $d(u, v)$, is the length of the shortest path connecting them. A graph G is said to be *connected* if there exists at least one path between each pair of distinct vertices of G. Otherwise, G is said to be *disconnected*.

The *diameter* of a connected graph, denoted by $\mathrm{diam}(G)$, is the greatest distance between two distinct vertices of the graph:

$$\mathrm{diam}(G) = \sup\{d(u, v) \mid u, v \in V,\ u \neq v\}.$$

Let U be a nonempty subset of the vertex set V. The *subgraph of G induced by U* is the graph consisting of the vertex set U and the edge set

$$\{\{x, y\} \in E \mid x, y \in U\}.$$

A subgraph of G is called a *connected component* of G if it is induced by a largest subset of vertices such that the subgraph is connected. The collection of vertex sets of all connected components of G forms a partition of V. The graph G is said to be *r-regular of order n* (or an *r-regular graph on n vertices*) if it has n vertices and each vertex is adjacent to exactly r other vertices.

Let $(L, [\cdot, \cdot])$ be a Lie algebra over a field $\mathbb{F}$. The *center* of L, denoted by $Z(L)$, is defined as

$$Z(L) := \{x \in L \mid [x, y] = 0,\ \forall y \in L\}.$$

For each $x \in L$, the *inner derivation* $\mathrm{ad}_x : L \to L$ is the linear map defined by

$$\mathrm{ad}_x(y) := [x, y].$$

Definition 2.1. ([14]) Let L be a Lie algebra over a field $\mathbb{F}$, with center $Z(L)$. The *commuting graph* of L, denoted by $\Gamma(L) = (V_L, E_L)$, is defined as the graph with:

$$V_L = L \setminus Z(L) \quad \text{and} \quad E_L = \{\{x, y\} \,|\, x \neq y \text{ and } [x, y] = 0\}.$$

Note that the commuting graph $\Gamma(L)$ of a Lie algebra L is finite if and only if the field $\mathbb{F}$ is finite, that is, $\mathbb{F} = \mathrm{F}_q$. Therefore, in the case $\mathbb{F} = \mathrm{F}_q$, the number of connected components $\mathrm{CC}_{\mathbb{F}_q}(L)$ of $\Gamma(L)$ is finite. We denote this number by $\left|\mathrm{CC}_{\mathbb{F}_q}(L)\right|$.

For convenience, in this paper, we use the notation $\mathrm{Lie}(n, k)$ to denote the class of solvable Lie algebras of dimension n whose derived algebra is k-dimensional and abelian, that is, $L^1 \cong \mathbb{F}^k$. Moreover, a vector space generated by the set $\{e_1, e_2, \ldots, e_k\}$ will be denoted by $\langle e_1, \ldots, e_k\rangle$. For any subspace S of a vector space, we define S^* as the set of all nonzero elements of S, that is,

$$S^* := S \setminus \{0\}.$$

For example, for any vector $x \in L$, we have

$$\langle x\rangle^* = \{\lambda x \mid \lambda \in \mathbb{F}, \lambda \neq 0\}.$$

3. Main results

In this section, we explicitly describe the commuting graphs associated with solvable Lie algebras whose abelian derived algebra of dimension 1 and codimension 1 and we study their connectivity. Recall that the problem is considered on an arbitrary field $\mathbb{F}$ of characteristic not equal to 2. We begin by stating the results for the case of decomposable Lie algebras of the form $L = L' \oplus \mathbb{F}^k$, where L' is a non-abelian Lie algebra and $\mathbb{F}^k$ is an abelian ideal.

Proposition 3.1. *Let $L = L' \oplus \mathbb{F}^k$ be a decomposable Lie algebra over a field $\mathbb{F}$. Then, we have the following assertions:*

(1) The set of vertices of $\Gamma(L)$ *is given by*

$$V_L = V_{L'} + \mathbb{F}^k := \{x' + a \mid x' \in V_{L'},\ a \in \mathbb{F}^k\}.$$

The set of edges of $\Gamma(L)$ *is given by*

$$E_L = \{\{x' + a,\ y' + b\} \mid \{x', y'\} \in E_{L'},\ a, b \in \mathbb{F}^k\}.$$

In particular, if $\mathbb{F} = \mathbb{F}_q$, *then*

$$|V_L| = |V_{L'}| \cdot q^k \quad \textit{and} \quad |E_L| = |E_{L'}| \cdot q^{2k}.$$

(2) The graph $\Gamma(L)$ *is connected if and only if* $\Gamma(L')$ *is connected. Moreover:*

- *If* $\Gamma(L)$ *is connected, then*
$$\operatorname{diam}(\Gamma(L)) = \operatorname{diam}(\Gamma(L')).$$
- *If* $\Gamma(L)$ *is disconnected, then*
$$\left|\mathrm{CC}_{\mathbb{F}_q}(L)\right| = \left|\mathrm{CC}_{\mathbb{F}_q}(L')\right|,$$
and each connected component $T \in \mathrm{CC}_{\mathbb{F}_q}(L)$ *can be written in the form*
$$T = T' + \mathbb{F}^k, \quad \textit{where } T' \in \mathrm{CC}_{\mathbb{F}_q}(L').$$

Proof. (1) By assumption, we have $L = L' \oplus \mathbb{F}^k$. Then the center satisfies

$$Z(L) = Z(L') + \mathbb{F}^k.$$

Thus, the vertex set is

$$V_L = V_{L'} + \mathbb{F}^k.$$

Moreover, for any vertices $x, y \in V_L$, we can write

$$x = x' + a, \quad y = y' + b, \quad \text{with } x', y' \in V_{L'},\ a, b \in \mathbb{F}^k.$$

Then,

$$[x, y] = 0 \iff [x', y'] = 0.$$

Hence, the edge set is

$$E_L = \left\{\{x' + a,\ y' + b\} \;\middle|\; \{x', y'\} \in E_{L'},\ a, b \in \mathbb{F}^k\right\}.$$

From this, when working over the field $\mathbb{F} = \mathbb{F}_q$, we obtain:

$$|V_L| = |V_{L'}| \cdot q^k \quad \text{and} \quad |E_L| = |E_{L'}| \cdot q^{2k}.$$

(2) By a similar argument to (1), consider a sequence of vertices $x_1, x_2, \ldots, x_m \in V_L$ where

$$x_i = x_i' + a_i, \quad x_i' \in L',\ a_i \in F^k, \quad \text{for } i = 1, \ldots, m.$$

Then the sequence $(x_1, x_2, \ldots, x_m)$ forms a path in $\Gamma(L)$ if and only if the sequence $(x_1', x_2', \ldots, x_m')$ forms a path of the same length in $\Gamma(L')$. From this, we obtain the following:

- If $\Gamma(L)$ is connected, then
$$\operatorname{diam}(\Gamma(L)) = \operatorname{diam}(\Gamma(L')).$$
- If $\Gamma(L)$ is disconnected, then
$$\left|\mathrm{CC}_{\mathbb{F}_q}(L)\right| = \left|\mathrm{CC}_{\mathbb{F}_q}(L')\right|,$$
and each connected component $T \in \mathrm{CC}_{\mathbb{F}_q}(L)$ can be expressed in the form
$$T = T' + \mathbb{F}^k, \quad \text{where } T' \in \mathrm{CC}_{\mathbb{F}_q}(L').$$

□

Remark 3.2. By Proposition 3.1, we only need to consider indecomposable Lie algebras.

We now consider the graphs of Lie algebras belonging to the classes $\mathrm{Lie}(n, 1)$ and $\mathrm{Lie}(n, n-1)$. Before stating the main results, we recall the classification of the class $\mathrm{Lie}(n, 1)$.

Proposition 3.3 ([12, Thm. 1])**.** *Let* L *be a real solvable Lie algebra whose derived algebra is one-dimensional. Then* L *is isomorphic to one of the following Lie algebras:*

(1) $\mathfrak{aff}(1) = \langle e_1, e_2 \rangle$, *where* $[e_1, e_2] = e_1$.

(2) $\mathfrak{h}_{2k+1} = \langle e_1, \ldots, e_k;\ e_{k+1}, \ldots, e_{2k}, e_{2k+1} \rangle, \quad [e_i, e_{k+i}] = e_{2k+1}, \quad \forall\, i \le k.$

(3) aff(1) $\oplus\mathbb{R}^{n-2}$; $\quad \mathfrak{h}_{2k+1} \oplus \mathbb{R}^{n-(2k+1)}$.

Remark 3.4. Although Proposition 3.3 is stated over the field of real numbers, it can be naturally generalized to any field $\mathbb{F}$ of characteristic not equal to 2. Specifically, in the case where L is nilpotent, the classification result of Khuhirun (see [10]) over fields of characteristic not equal to 2 shows that such Lie algebras consist solely of Heisenberg Lie algebras and their trivial extensions by abelian Lie algebras. In the case where L is solvable but not nilpotent, an inductive argument on the dimension, using the known classification for $n \leq 6$, similarly to Schöbel's method, shows that the resulting Lie algebras are limited to $aff(1)$ and their trivial extensions.

Using the above results, we obtain the following conclusion concerning the commuting graphs of Lie algebras in the class $\mathrm{Lie}(n, 1)$.

Proposition 3.5. *Let $L \in \mathrm{Lie}(n, 1)$ be an indecomposable Lie algebra. Then:*

(1) If $n \leq 3$, then $\Gamma(L)$ is disconnected; moreover, each connected component of $\Gamma(L)$ is a complete graph. In particular, if $\mathbb{F} = \mathbb{F}_q$, then the number of connected components of $\Gamma(L)$, as well as the number of vertices and edges in each component are determined as follows:

Lie bracket	$Z(L)$	$CC_F(L)$	$\lvert V_K\rvert$	$\lvert E_K\rvert$	$\lvert CC_{F_q}(L)\rvert$
$[e_1, e_2] = e_1$	$\{0\}$	$K_{\langle e_1\rangle^*}, \quad K_{\langle \mu e_1+e_2\rangle^*} \quad (\mu \in \mathbb{F})$	$q-1$	$\binom{q-1}{2}$	$q+1$
$[e_1, e_2] = e_3$	$\langle e_3\rangle$	$K_{\langle e_1\rangle^*+\langle e_3\rangle}, \ K_{\langle \mu e_1+e_2\rangle^*+\langle e_3\rangle} \ (\mu \in \mathbb{F})$	q^2-q	$\binom{q^2-q}{2}$	$q+1$

(2) If $n > 3$, then $\Gamma(L)$ is connected, and moreover, its diameter satisfies $\mathrm{diam}(\Gamma(L)) = 2$. *In particular, over the finite field $\mathbb{F} = \mathbb{F}_q$, the graph $\Gamma(L)$ is a $(q^{n-1} - q - 1)$-regular graph with the number of vertices and edges given by:*

$$|V_L| = q(q^{n-1} - 1), \quad |E_L| = \frac{q^{2n-1} - q^{n+1} - 2q^n + q^2 + q}{2}.$$

To prove Proposition 3.5, we need the following lemma:

Lemma 3.6. *Let $L \in \mathrm{Lie}(n, 1)$ be an indecomposable n-dimensional Lie algebra over the finite field $\mathbb{F}_q$. Then the number of edges in the commuting graph $\Gamma(L)$ is given by:*

$$|E_L| = \frac{1}{2}\left[q^{2n-1} - (q+1)q^{n+\dim Z(L)-1} + q^{2\dim Z(L)} - q^n + q^{\dim Z(L)}\right].$$

Proof. We have

$$L \times L = [V_L \sqcup Z(L)] \times [V_L \sqcup Z(L)] = (V_L \times V_L) \sqcup (V_L \times Z(L)) \sqcup (Z(L) \times V_L) \sqcup (Z(L) \times Z(L)),$$

where the unions are disjoint. Let

$C(L) = \{(x, y) \in L \times L \mid [x, y] = 0\}$, $\quad$ (note that $(x, y) \neq (y, x)$),
$C(V_L) = \{(x, y) \in V_L \times V_L \mid [x, y] = 0\}$,
$C_L(x) = \{y \in L \mid [x, y] = 0\}$.
It follows that $|C(L)| = \sum_{x\in L} |C_L(x)|$, and $|E_L| = \frac{|C(V_L)|-|V_L|}{2}$.
From this, we obtain:

$$|C(L)| = |C(V_L)| + 2|Z(L)| \cdot |V_L| + |Z(L)|^2 = |C(V_L)| + 2|Z(L)| \cdot |L| - |Z(L)|^2,$$

and

$$|E_L| = \frac{|C(V_L)| - |V_L|}{2} = \frac{|C(L)| - 2|Z(L)| \cdot |L| + |Z(L)|^2 - |L| + |Z(L)|}{2}.$$

On the other hand, since $\dim L^1 = 1$, we have

$$|C(L)| = \sum_{x\in L} |C_L(x)| = |L| \cdot |Z(L)| + \sum_{x\in L\setminus Z(L)} |C_L(x)| = |L| \cdot |Z(L)| + |L| \sum_{x\in L\setminus Z(L)} \frac{1}{\mathrm{Im}(\mathrm{ad}_x)}.$$

Note that:

$$|L| = q^n, \quad |Z(L)| = q^{\dim Z(L)}, \quad \mathrm{Im}(\mathrm{ad}_x) = q, \quad \forall x \in L \setminus Z(L).$$

Substituting into the above formula, we obtain:

$$|C(L)| = |L| \cdot |Z(L)| + |L| \cdot \frac{|L| - |Z(L)|}{q} = q^{n+\dim Z(L)} + q^{n-1}\left(q^n - q^{\dim Z(L)}\right).$$

Therefore,

$$\begin{aligned} |E_L| &= \frac{1}{2q}\Big[q^{2n} - q^{n+\dim Z(L)} - q^{n+1+\dim Z(L)} + q^{2\dim Z(L)+1} - q^{n+1} + q^{1+\dim Z(L)}\Big] \\ &= \frac{1}{2}\Big[q^{2n-1} - (q+1)q^{n+\dim Z(L)-1} + q^{2\dim Z(L)} - q^n + q^{\dim Z(L)}\Big]. \end{aligned}$$

□

Proof of Proposition 3.5. By assumption, $L \in \mathrm{Lie}(n,1)$ is an indecomposable Lie algebra. We then consider the following two mutually exclusive cases:

(1) If $n \leq 3$, then $L \cong aff(1)$ or $L \cong \mathfrak{h}_3$ (the 3-dimensional Heisenberg Lie algebra). We consider each case separately as follows:

Case 1: $L \cong aff(1)$. In this case, $Z(L) = \{0\}$ and $V_L = L \setminus \{0\}$.

For all $x, y \in V_L$, suppose $x = a_1e_1 + a_2e_2$ and $y = b_1e_1 + b_2e_2$ with $(a_1, a_2), (b_1, b_2) \neq (0,0)$. Then

$$[x,y] = 0 \iff a_1b_2 - a_2b_1 = 0,$$

that is, $y \in \langle x \rangle^*$. It follows that:

- $\Gamma(L)$ is disconnected, since for any vertex $x \in V_L$, no vertex in $\langle x \rangle^*$ can be connected to any vertex outside $\langle x \rangle^*$;
- Each connected component containing a vertex x is a complete graph of the form $K_{\langle x \rangle^*}$.

Then,

$$\mathrm{CC}_{\mathbb{F}}(L) = \left\{K_{\langle e_1 \rangle^*},\ K_{\langle \mu e_1 + e_2 \rangle^*} \mid \mu \in \mathbb{F}\right\},$$

where $\mathrm{CC}_{\mathbb{F}}(L)$ denotes the set of connected components of $\Gamma(L)$ over the field $\mathbb{F}$.

Therefore, when working over the finite field $\mathbb{F}_q$, we have:

- Each complete graph $K_{\langle x \rangle^*}$, for $x \in V_L$, has the same number of vertices and edges, given by:

$$|V_K| = q - 1, \quad |E_K| = \binom{q-1}{2}.$$

- The number of connected components is

$$|\mathrm{CC}_{\mathbb{F}_q}(L)| = q + 1.$$

Case 2: $L \cong \mathfrak{h}_3$. In this case, $Z(L) = \langle e_3 \rangle$ and $V_L = L \setminus \langle e_3 \rangle$. For all $x, y \in V_L$, suppose

$$x = a_1e_1 + a_2e_2 + a_3e_3 \quad \text{with } (a_1, a_2) \neq (0,0),$$

$$y = b_1e_1 + b_2e_2 + b_3e_3 \quad \text{with } (b_1, b_2) \neq (0,0).$$

Then

$$[x,y] = 0 \iff a_1b_2 - a_2b_1 = 0.$$

By a similar argument, we also deduce that:

- $\Gamma(L)$ is disconnected, since for any vertex $x = a_1e_1 + a_2e_2 + a_3e_3 \in V_L$, no vertex in $\langle a_1e_1 + a_2e_2 \rangle^* + \langle e_3 \rangle$ can be connected to any vertex outside $\langle a_1e_1 + a_2e_2 \rangle^* + \langle e_3 \rangle$. In other words, the connected component containing x is a complete graph of the form $K_{\langle a_1e_1 + a_2e_2 \rangle^* + \langle e_3 \rangle}$.
- It follows that

$$\mathrm{CC}_{\mathbb{F}}(L) = \left\{K_{\langle e_1 \rangle^* + \langle e_3 \rangle},\ K_{\langle \mu e_1 + e_2 \rangle^* + \langle e_3 \rangle} \mid \mu \in \mathbb{F}\right\},$$

where $\mathrm{CC}_{\mathbb{F}}(L)$ denotes the set of connected components of $\Gamma(L)$ over the field $\mathbb{F}$.

Therefore, when working over the finite field $\mathbb{F}_q$, we have:

- Each connected component has the same number of vertices and edges, given by:

$$|V_K| = q^2 - q, \quad |E_K| = \binom{q^2-q}{2}.$$

- The number of connected components is

$$|\mathrm{CC}_{\mathbb{F}_q}(L)| = q + 1.$$

(2) If $n > 3$, then for every $x \in V_L$, we consider the derivation operator:

$$\mathrm{ad}_x : L \to L, \quad \mathrm{ad}_x(y) = [x, y].$$

We always have:

$$\dim(\mathrm{Im}(\mathrm{ad}_x)) = 1.$$

Since:

$$\dim L = \dim(\ker(\mathrm{ad}_x)) + \dim(\mathrm{Im}(\mathrm{ad}_x)),$$

it follows that:

$$\dim(\ker(\mathrm{ad}_x)) = n - 1.$$

Therefore, for all $x, y \in V_L$, we have:

$$\dim\left(\ker(\mathrm{ad}_x) \cap \ker(\mathrm{ad}_y)\right) \geq n - 2 \geq 2.$$

So, there always exists a vertex $z \in \ker(\mathrm{ad}_x) \cap \ker(\mathrm{ad}_y)$. In other words, $d(x, y) \leq 2$. On the other hand, since L is non-abelian, there obviously exist two vertices $x', y' \in V_L$ such that $[x', y'] \neq 0$, or equivalently $d(x', y') \geq 2$.

Hence, we conclude that:

$$\mathrm{diam}(\Gamma(L)) = 2.$$

- When $\mathbb{F} = \mathbb{F}_q$, the vertex set of the graph $\Gamma(L) = \Gamma(h_{2k+1})$ is given by:

$$V_{h_{2k+1}} = \left\{ \sum_{i=1}^{2k+1} a_i e_i \;\middle|\; a_i \in \mathbb{F}_q \right\} \setminus \langle e_{2k+1} \rangle.$$

This graph has a total of $q(q^{2k} - 1)$ vertices. Now, for each arbitrary vertex $x \in V_{h_{2k+1}}$, we prove that x is adjacent to exactly $q^{2k} - q - 1$ distinct vertices in $V_{h_{2k+1}}$. Indeed, suppose:

$$x = \sum_{i=1}^{2k+1} a_i e_i.$$

Since $x \in V_{h_{2k+1}}$, the coefficients $a_1, \ldots, a_{2k}$ cannot all be zero simultaneously.
Let

$$y = \sum_{i=1}^{2k+1} b_i e_i$$

be any vertex in $V_{h_{2k+1}}$. Then we can easily verify that:

$$[x, y] = 0 \iff \left[\sum_{i=1}^{2k+1} a_i e_i, \sum_{i=1}^{2k+1} b_i e_i \right] = 0 \iff \sum_{i=1}^{k} (a_i b_{k+i} - a_{k+i} b_i) = 0.$$

We treat a_i, a_{k+i} as constants and b_i, b_{k+i} as variables (for $i = 1, \ldots, k$). Then equation

$$\sum_{i=1}^{k} (a_i b_{k+i} - a_{k+i} b_i) = 0$$

is a linear equation with variables $2k$ on the field $\mathbb{F}_q$, and since the coefficients are not all zero, the number of non-trivial solutions is $q^{2k-1} - 1$. It follows that the number of vertices adjacent to x is:

$$(q^{2k-1} - 1)q - 1 = q^{2k} - q - 1,$$

where we subtract one to exclude x itself. In other words, $\Gamma(h_{2k+1})$ is a $(q^{2k} - q - 1)$-regular graph on $q(q^{2k} - 1)$ vertices.
Finally, the number of edges in $\Gamma(h_{2k+1})$ is directly derived from Lemma 3.6.

□

Combining the results from Proposition 3.5 and Lemma 3.6, we obtain the following formulas for the number of edges and vertices of the commuting graph of indecomposable Lie algebras in the class $\mathrm{Lie}(n,1)$ in the finite field $\mathbb{F}_q$.

Remark 3.7. Let $L \in \mathrm{Lie}(n,1)$ be an indecomposable Lie algebra over the finite field $\mathbb{F}_q$.

(1) If $n = 2$, then the number of vertices is $|V_L| = q^2 - 1$, and the number of edges is
$$|E_L| = \frac{q^3 - 2q^2 - q + 2}{2}.$$
(2) If $n \geq 3$, then the number of vertices is $|V_L| = q^n - q$, and the number of edges is
$$|E_L| = \frac{q^{2n-1} - q^{n+1} - 2q^n + q^2 + q}{2}.$$

In the following, we consider the graph associated with the class $\mathrm{Lie}(n, n-1)$ of n-dimensional Lie algebras whose derived algebra is abelian and of dimension $n-1$, that is, $L^1 \cong \mathbb{F}^{n-1}$. Note that, according to Graaf (2005) (see [9]), each Lie algebra $L \in \mathrm{Lie}(n, n-1)$ can always be expressed in the following form:
$$L = \langle e_n \rangle \oplus_d L^1, \quad \text{where } L^1 = \langle e_1, \ldots, e_{n-1} \rangle,$$
in which $d : L^1 \to L^1$ is a derivation on L^1 defined by $d(x) = [e_n, x]$ for all $x \in L^1$. In other words, L can always be viewed as a one-dimensional extension of L^1 by a derivation d.

Below is an explicit description of the commuting graph of Lie algebras in the class $\mathrm{Lie}(n, n-1)$.

Proposition 3.8. *Let $L \in \mathrm{Lie}(n, n-1)$. Then the graph $\Gamma(L)$ is disconnected, and each connected component is a complete graph of the following form:*
$$\mathrm{CC}_F(L) = \left\{ K_{(L^1)^*},\ K_{\langle \mu_1 e_1 + \cdots + \mu_{n-1} e_{n-1} + e_n \rangle^*} \;\middle|\; \mu_i \in \mathbb{F} \right\}.$$
In particular, if $\mathbb{F} = \mathbb{F}_q$, then the number of connected components of $\Gamma(L)$ is $q^{n-1} + 1$, and the number of vertices and edges in each component is given in the following:

$CC_F(L)$	$\lvert V_K \rvert$	$\lvert E_K \rvert$	$\lvert \mathrm{CC}_{\mathbb{F}_q}(L) \rvert$
$K_{\langle \mu_1 e_1 + \cdots + \mu_{n-1} e_{n-1} + e_n \rangle^*}, \quad \mu_i \in F$	$q-1$	$\binom{q-1}{2}$	q^{n-1}
$K_{(L^1)^*}$	$q^{n-1} - 1$	$\binom{q^{n-1}-1}{2}$	1

Proof. By assumption, $L \in \mathrm{Lie}(n, n-1)$ and $L^1 \cong \mathbb{F}^{n-1}$. Then the derivation $d : L^1 \to L^1$ is an isomorphism, which implies that L is indecomposable and $Z(L) = \{0\}$. Hence, the vertex set of the commuting graph is:
$$V_L = L \setminus \{0\}.$$
For all elements $ae_n + x_1,\ be_n + y_1 \in V_L$, with $a, b \in \mathbb{F}$ and $x_1, y_1 \in L^1$, we have:
$$[ae_n + x_1,\ be_n + y_1] = 0 \iff d(ay_1 - bx_1) = 0.$$
Since d is an isomorphism, this implies:
$$ay_1 - bx_1 = 0.$$
Thus, the condition $[ae_n + x_1,\ be_n + y_1] = 0$ is equivalent to one of the following:

- $x_1 = y_1 = 0$;
- $a = b = 0$;
- $y_1 \in \langle x_1 \rangle$.

It follows that $\Gamma(L)$ is disconnected. Moreover, we have
$$\mathrm{CC}_F(L) = \left\{ K_{(L^1)^*},\ K_{\langle \mu_1 e_1 + \cdots + \mu_{n-1} e_{n-1} + e_n \rangle^*} \;\middle|\; \mu_i \in \mathbb{F} \right\}.$$
In particular, if $\mathbb{F} = \mathbb{F}_q$, then
$$|\mathrm{CC}_F(L)| = q^{n-1} + 1.$$
This includes:

- q^{n-1} connected components of the form $K_{\langle \mu_1 e_1 + \cdots + \mu_{n-1} e_{n-1} + e_n \rangle^*}$, each containing $q-1$ vertices and $\binom{q-1}{2}$ edges;
- one connected component $K_{(L^1)^*}$ containing $q^{n-1} - 1$ vertices and $\binom{q^{n-1}-1}{2}$ edges.

□

4. Conclusion

This paper provides an explicit and detailed description of the commuting graph $\Gamma(L)$ associated with the entire class of solvable Lie algebras L whose derived ideal L^1 is abelian and has dimension one or codimension one. The study thoroughly analyzed the properties of connectivity and diameter of these graphs, revealing that only the Heisenberg Lie algebras and their trivial extensions ($\dim > 3$) possess a connected commuting graph with a diameter of $\operatorname{diam}(\Gamma(L)) = 2$; while all other cases are disconnected, with the structure of each connected component being precisely described as complete subgraphs.

The core contribution of this work lies in extending the research object in an early-stage field. While the pioneering work by Wang and Xia ([14]) laid the foundation by focusing on the connectivity of Simple Lie Algebras - which are relatively simple in structure and completely classified - our research is one of the first and most comprehensive efforts to apply Graph Theory to Solvable Lie Algebras. This class of algebras has a more complex structure and remains unclassified. We not only determined the diameter when the graph is connected but also provided an explicit geometric description of all connected components when the graph is disconnected.

For future work, we propose extending the investigation of $\Gamma(L)$ to other unexplored classes of Lie algebras, including: the class of solvable Lie algebras whose derived ideal is abelian and two-dimensional, the class of nilpotent Lie algebras of class two, and the class of Semisimple Lie Algebras. This extension aims to consolidate the bridge between algebraic structure and graph properties.

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