COMPARISION AND STUDY OF NUMERICAL METHODS FOR DYNAMIC RESPONSE EVALUATION OF SDOF

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Abstract. Written for senior-year undergraduates and first-year graduate students with solid backgrounds in differential and integral calculus, this paper is oriented toward engineers and applied mathematicians. Consequently, this paper should be useful to senior-year undergraduates the finite element method [1]. The scaled direct approach is adopted for this purpose and each step in the finite element solution process is given in full detail. For this reason, all students must be exposed to (and indeed should master). This paper provides the general framework for the development of nearly all (nonstructural) finite element models. The finite element method of analysis is a very powerful, modern computational tool. Applications range from deformation and stress analysis of automotive, aircraft, building, and bridge structures to field analysis of beat flux, fluid flow, magnetic flux, seepage, and other flow problems.
This paper presents study and comparison of numerical methods which are used for evaluation of dynamic response. A Single Degree of Freedom (SDF)-linear problem is solved by means of Newmark's Average acceleration method [2], Linear acceleration method [2], Central Difference method [6,7] with the help of MATLAB. The advantages, disadvantages, relative precision and applicability of these numerical methods are discussed throughout the analysis.
Keywords. Finite element method, central difference method, Newmark's constant average acceleration method, Newmark's linear acceleration method.

1 INTRODUCTION

The basic idea behind the finite element method is to divide the structure, body, or region being analyzed into a large number of finite elements, or simply elements. These elements may be one, two, or three dimensional.

In 1941, Alexander Hrennikoff (*was born in Russia, graduated from the Institute of Railway Engineering in Moscow*) he developed the lattice analogy which models membrane and plate bending of structures as a lattice framework [1].

In the early 1960s, engineers used the method for approximate solution of problems in stress analysis, fluid flow, heat transfer. A book by Argyris in 1955 on energy theorems and matrix methods laid a foundation for further developments in finite element studies. In 1956, Turner et al derived stiffness matrices for truss beam and other elements.

Today, the developments in mainframe computers and availability of powerful microcomputers has brought this method within reach of students and engineers working in industries.

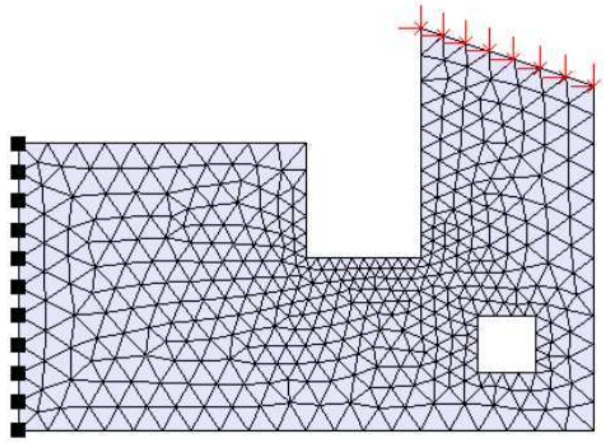

Figure 1: FEM mesh created by an analyst prior to finding a solution to a magnetic problem using FEM software.

2 PROBLEM SOLVING METHODOLOGY

Basic steps of finite element analysis [9]

Step 1 - Discretization of real continuum or structure
Step 2 - Identify primary unknown quantity
Step 3 - Interpolation functions and the derivation of Interpolation functions
Step 4 - Derivation of Element equation
Step 5 - Derive overall Stiffness Equation

The static equilibrium equation:

$$[K]\{q\}=\{R\} \tag{2.1}$$

Dynamic equilibrium equations

$$[M]\{\ddot{q}(t)\}+[C]\{\dot{q}(t)\}+[K]\{q(t)\}=\{R(t)\} \tag{2.2}$$

where:

$[K]$, $[M]$, $[C]$ are overall stiffness matrix, mass matrix, damping matrix;

$\{q(t)\}$, nodal force vector $\{\dot{q}(t)\}$, $\{\ddot{q}(t)\}$ - vector displacement node, velocity vector, acceleration vector (dynamics problems)

$\{R\}$ - nodal force vector (static problem)

$\{R(t)\}$ - Nodal applied force vector (dynamic problem)

$$[K']_e=[T]_e^T[K]_e[T]_e \tag{2.3}$$

$$[M']_e=[T]_e^T[M]_e[T]_e \tag{2.4}$$

which $[T]_e$ is transfer matrix:

$$[T]_e=\begin{bmatrix} [L] & & . & [0] \\ & [L] & . & \\ . & . & . & . \\ [0] & & . & [L] \end{bmatrix}$$

and $[L]$ is the cosine of a matrix:

$$[L]=\begin{bmatrix} l_{xx'} & m_{xy'} & n_{xz'} \\ l_{yx'} & m_{yy'} & n_{yz'} \\ l_{zx'} & m_{zy'} & n_{zz'} \end{bmatrix}=\begin{bmatrix} cos(x,x') & cos(x,y') & cos(x,z') \\ cos(y,x') & cos(y,y') & cos(y,z') \\ cos(z,x') & cos(z,y') & cos(z,z') \end{bmatrix}$$

where

$l_{xx'}$, $m_{xy'}$, $n_{xz'}$ - is the direction cosine of x axis;

$l_{yx'}$, $m_{yy'}$, $n_{yz'}$ - is the direction cosine of y axis

$l_{zx'}$, $m_{zy'}$, $n_{zz'}$ - is the direction cosine of z axis

Global stiffness matrix can be expressed as

$$[K]_e=\int_V [D]_e^T [E_0]_e [D]_e \, dV \tag{2.5}$$

where:

$[D]_e$ - matrix deformation - displacement is derived from the relationship deformation-displacement

$$\{\varepsilon\}_e=[D]_e\{q\}_e=[\partial][B]_e\{q\}_e \tag{2.6}$$

$$[D]_e=[\partial][B]_e \tag{2.7}$$

$[\partial]$ - The differential operator matrix is determined from elastic theory

$[B]_e$ - The function matrix in the displacement function expression $\{U\}_e=[B]_e\{q\}_e$, This function represents the variation of the displacement within the element.

$[E_0]_e$ - is the elastic compliance matrix, and is symmetric. The general form of the strain–stress relation $\{\sigma\}_e=[E_0]_e\{\varepsilon\}_e$

Mass matrix of elements in the local coordinate system

$$[M]_e=\int_V [B]_e^T \rho [B]_e \, dV \tag{2.8}$$

ρ - The density of a material is defined as its mass per unit volume

$[C]$ - is damping matrix of the system is combined linearly from the stiffness matrix $[K]$ and mass matrix $[M]$ can be determined by the formula:

$$[C]=\alpha[M]+\beta[K]$$

with

α is the mass proportional Rayleigh damping coefficient

β is the stiffness proportional Rayleigh damping coefficient

Classical Rayleigh damping results in different damping ratios for different response frequencies, according to the equation:

$$\xi = \frac{\alpha + \beta\omega^2}{2\omega} \quad (2.9)$$

ξ is the damping ratio (a value of 1 corresponds to critical damping)

ω is the response frequency in rad/s.

It can be seen from this that the mass proportional term gives a damping ratio inversely proportional to response frequency and the stiffness proportional term gives a damping ratio linearly proportional to response frequency.

Nodal force vector: $\{R^{'}\}_e = [T]_e^T \{R\}_e$ (2.10)

Dynamic load changes with time $f(t)$:

$$\{R^{'}(t)\}_e = [T]_e^T \{R\}_e f(t) \quad (2.11)$$

Nodal force vector of elements in the local coordinate system

$$\{R\}_e = \{R_V\}_e + \{R_S\}_e + \{R_{\varepsilon^o}\}_e \quad (2.12)$$

where:

$$\{R_V\}_e = \int_V [B]_e^T \{p_V\} dV$$

$$\{R_S\}_e = \int_S [B]_e^T \{p_S\} dS$$

$$\{R_{\varepsilon^o}\}_e = \int_V [D]_e^T [E_o]_e \{\varepsilon^o\}_e dV$$

V , S - volume and area of the element;

$\{p_V\} = \{p_{Vx} \quad p_{Vy} \quad p_{Vz}\}^T$ - force distributed by volume

$\{p_S\} = \{p_{Sx} \quad p_{Sy}\}^T$ - force distributed by area

$\{\varepsilon^0\}_e = \{\varepsilon_x^0 \quad \varepsilon_y^0 \quad \varepsilon_z^0 \quad \gamma_{xy}^0 \quad \gamma_{zy}^0 \quad \gamma_{xz}^0\}^T$ - Initial forced distortion in the element

Step 6 – Solve for the unknowns

Step 7 – Interpretation of results

A differential equation is used for calculating structure by Finite Element Method. It is tough to solve this equation by analytical method

2.1. The central difference method [6]

Step 1 - Initial calculations

$$\{\ddot{q}\}_0 = \frac{\{R\}_0 - [C]\{\dot{q}\}_0 - [K]\{q\}_0}{[M]} \quad (2.13)$$

$$\{q\}_{-1} = \{q\}_0 - \Delta t\{\dot{q}\}_0 + \frac{(\Delta t)^2}{2}\{\ddot{q}\}_0 \quad (2.14)$$

$$[\hat{K}] = \frac{[M]}{(\Delta t)^2} + \frac{[C]}{2\Delta t} \quad (2.15)$$

$$[a] = \frac{[M]}{(\Delta t)^2} - \frac{[C]}{2\Delta t} \quad (2.16)$$

$$[b]=[K]-\frac{2[M]}{(\Delta t)^2} \quad (2.17)$$

Step 2 - Calculate the values with each time step

$$\{\hat{R}\}_i=\{R\}_i-[a]\{q\}_{i-1}-[b]\{q\}_i \quad (2.18)$$

$$\{q\}_{i+1}=\frac{\{\hat{R}\}_i}{[\hat{K}]} \quad (2.19)$$

$$\{\dot{q}\}_i=\frac{\{q\}_{i+1}-\{q\}_{i-1}}{2\Delta t} \quad (2.20)$$

$$\{\ddot{q}\}_i=\frac{\{q\}_{i+1}-2\{q\}_i+\{q\}_{i-1}}{(\Delta t)^2} \quad (2.21)$$

Step 3: Calculate for next time steps

- Example 1: Determine the displacement of the single degree of freedom system subjected to dynamic loads for the figure 2

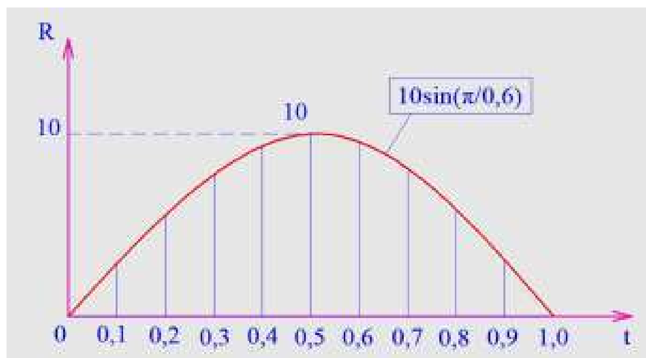

Figure 2: A load changes with time.

- Solution:

The mass of the object: $m=0,2533$, stiffness $k=10$, Damping Coefficient $c=0,1592$. the initial parameters: At the beginning of the survey $t=0$; initial displacement and velocity $q_0=0$; $\dot{q}_0=0$;

Step 1:

$$\ddot{q}_0=\frac{R_0-c.\dot{q}_0-k.q_0}{m}=0$$

$$q_{-1}=q_0-(\Delta t)\dot{q}_0+\frac{(\Delta t)^2}{2}\ddot{q}_0=0$$

$$\hat{k}=\frac{m}{(\Delta t)^2}+\frac{c}{2\Delta t}=26,13$$

$$a=\frac{m}{(\Delta t)^2}-\frac{c}{2\Delta t}=24,53$$

$$b = k - \frac{2m}{(\Delta t)^2} = -40,66$$

Step 2: Calculated for each time step:

$$\widehat{R}_i = R_i - a.q_{i-1} - b.q_i = R_i - 24,53.q_{i-1} - 40,66.q_i$$

$$q_{i+1} = \frac{\widehat{R}_i}{\widehat{k}} = \frac{\widehat{R}_i}{26,13}$$

Step 3:

Table 1: Calculate for next time steps (numerical results show in table results).

t_i	R_i	q_{i-1}	q_i	$\widehat{R}_i$	q_{i+1}
0,0	0,0000	0,0000	0,0000	0,0000	0,0000
0,1	5,0000	0,0000	0,0000	5,0000	0,1914
0,2	8,6602	0,0000	0,1914	16,4419	0,6293
0,3	10,0000	0,1914	0,6293	30,8934	1,1825
0,4	8,6602	0,6293	1,1825	41,3001	1,5808
0,5	5,0000	1,1825	1,5808	40,2649	1,5412
0,6	0,0000	1,5808	1,5412	23,8809	0,9141
0,7	0,0000	1,5412	0,9141	-0,6456	-0,0247
0,8	0,0000	0,9141	-0,0247	-23,4309	-0,8968
0,9	0,0000	-0,0247	-0,8968	-35,8598	-1,3726
1,0	0,0000	-0,8968	-1,3726	-33,8058	-1,2940

2.2. Newmark's constant average acceleration method [9]

Step 1:

$$\ddot{q}_0 = \frac{R_0 - c.\dot{q}_0 - k_0.q_0}{m} = 0$$

The time step is the incremental change in time $\Delta t = 0,1\text{sec}$

$$\widehat{k} = k + \frac{2}{\Delta t}c + \frac{4}{(\Delta t)^2}m = 114,5$$

$$a = \frac{4}{\Delta t}m + 2.c = 10,45;\ b = 2m = 0,5066$$

Step 2: Calculated for each time step:

$$\Delta\widehat{R}_i = \Delta R_i + a.\dot{q}_i + b.\ddot{q}_i = \Delta R_i + 10.45\dot{q}_i + 0.5066\ddot{q}_i$$

$$\Delta q_i = \frac{\Delta\widehat{R}_i}{\widehat{k}_i} = \frac{\Delta\widehat{R}_i}{114,5}$$

$$\Delta\dot{q}_i = \frac{2}{\Delta t}\Delta q_i - 2\dot{q}_i = 20\Delta q_i - 2\dot{q}_i$$

$$\Delta\ddot{q}_i = \frac{4}{(\Delta t)^2}\left(\Delta q_i - \Delta t.\dot{q}_i\right) - 2.\ddot{q}_i = 400\left(\Delta q_i - 0,1\dot{q}_i\right) - 2.\ddot{q}_i$$

$$q_{i+1} = q_i + \Delta q_i;\ \dot{q}_{i+1} = \dot{q}_i + \Delta\dot{q}_i;\ \ddot{q}_{i+1} = \ddot{q}_i + \Delta\ddot{q}_i$$

Step 3:

Table 2: Calculate for next time steps (numerical results show in table results).

t_i	R_i	$\ddot{q}_i$	ΔR_i	$\Delta\widehat{R}_i$	Δq_i	$\Delta\dot{q}_i$	$\Delta\ddot{q}_i$	$\dot{q}_i$	q_i
0,0	0,0000	0,0000	5,0000	5,0000	0,0437	0,8733	17,4666	0,0000	0,0000
0,1	5,0000	17,4666	3,6603	21,6356	0,1890	2,0323	5,7137	0,8733	0,0437
0,2	8,6602	23,1803	1,3398	43,4485	0,3794	1,7776	-10,8087	2,9057	0,2326
0,3	10,0000	12,3724	-1,3397	53,8708	0,4705	0,0428	-23,8893	4,6833	0,6121
0,4	8,6602	-11,5169	-3,6602	39,8948	0,3484	-2,483	-26,6442	4,7261	1,0825
0,5	5,0000	-38,1611	-5,0000	-0,9009	-0,0079	-4,641	-16,5122	2,2422	1,4309
0,6	0,0000	-54,6733	0,0000	-52,7740	-0,4609	-4,418	20,9716	-2,3995	1,4231
0,7	0,0000	-33,7017	0,0000	-88,3275	-0,7714	-1,791	31,5787	-6,8133	0,9622
0,8	0,0000	-2,1229	0,0000	-91,0486	-0,7952	1,3159	30,5646	-8,6095	0,1908
0,9	0,0000	28,4417	0,0000	-61,8123	-0,5398	3,7907	18,9297	-7,2936	-0,6044
1,0	0,0000	47,3714	0,0000					-3,5029	-1,1442

2.3. Newmark's linear acceleration method [9]

Step 1:

$$\ddot{q}_0 = \frac{R_0 - c.\dot{q}_0 - k_0.q_0}{m} = 0$$

$$\Delta t = 0,1\text{sec}$$

$$\widehat{k} = k + \frac{3}{\Delta t}c + \frac{6}{(\Delta t)^2}m = 166,8$$

$$a = \frac{6}{\Delta t}m + 3.c = 15,68\,;\; b = 3m + \frac{\Delta t}{2}c = 0,7679$$

Step 2:

$$\Delta\widehat{R}_i = \Delta R_i + a.\dot{q}_i + b.\ddot{q}_i = \Delta R_i + 15,68\dot{q}_i + 0,7679\ddot{q}_i$$

$$\Delta q_i = \frac{\Delta\widehat{R}_i}{\widehat{k}_i} = \frac{\Delta\widehat{R}_i}{166,8}$$

$$\Delta\dot{q}_i = \frac{3}{\Delta t}\Delta q_i - 3\dot{q}_i - \frac{\Delta t}{2}\ddot{q}_i = 30\Delta q_i - 3\dot{q}_i - 0,05\ddot{q}_i$$

$$\Delta\ddot{q}_i = \frac{6}{(\Delta t)^2}\left(\Delta q_i - \Delta t.\dot{q}_i\right) - 3.\ddot{q}_i = 600\left(\Delta q_i - 0,1\dot{q}_i\right) - 3.\ddot{q}_i$$

$$q_{i+1} = q_i + \Delta q_i\,;\; \dot{q}_{i+1} = \dot{q}_i + \Delta\dot{q}_i\,;\; \ddot{q}_{i+1} = \ddot{q}_i + \Delta\ddot{q}_i$$

Step 3:

Table 3: Calculate for next time steps (numerical results show in table results).

t_i	R_i	$\ddot{q}_i$	ΔR_i	$\Delta \widehat{R}_i$	Δq_i	$\Delta \dot{q}_i$	$\Delta \ddot{q}_i$	$\dot{q}_i$	q_i
0,0	0,0000	0,0000	5,0000	5,0000	0,0300	0,8995	17,9903	0,0000	0,0000
0,1	5,0000	17,9903	3,6603	31,5749	0,1893	2,0824	5,6666	0,8995	0,0300
0,2	8,6602	23,6569	1,3398	66,2479	0,3973	1,7897	-11,5191	2,9819	0,2193
0,3	10,0000	12,1378	-1,3397	82,7784	0,4964	-0,0296	-24,8677	4,7716	0,6166
0,4	8,6602	-12,7299	-3,6602	60,8987	0,3652	-2,6336	-27,2127	4,7420	1,1130
0,5	5,0000	-39,9426	-5,0000	-2,6205	-0,0157	-4,7994	-16,1033	2,1084	1,4782
0,6	0,0000	-56,0459	0,0000	-85,2198	-0,5110	-4,4558	22,9749	-2,6911	1,4625
0,7	0,0000	-33,0710	0,0000	-137,426	-0,8241	-1,6292	33,5584	-7,1469	0,9514
0,8	0,0000	0,4874	0,0000	-137,196	-0,8227	1,6218	31,4613	-8,7761	0,1273
0,9	0,0000	31,9487	0,0000	-87,6156	-0,5254	4,1031	18,1644	-7,1543	-0,6954
1,0	0,0000	50,1130	0,0000					-3,0512	-1,2208

3 COMPUTATIONAL RESULTS

Table 4: Compare results of displacement values between methods with theoretical results.

t_i	Central Difference Method	Newmark's constant average acceleration method	Newmark's linear acceleration method	Analytical results based on theory
0,0	0,0000	0,0000	0,0000	0.0000
0,1	0,1914	0,0437	0,0300	0,0328
0,2	0,6293	0,2326	0,2193	0,2332
0,3	1,1825	0,6121	0,6166	0,6487
0,4	1,5808	1,0825	1,1130	1,1605
0,5	1,5412	1,4309	1,4782	1,5241
0,6	0,9141	1,4231	1,4625	1,4814
0,7	-0,0247	0,9622	0,9514	0,9245
0,8	-0,8968	0,1908	0,1273	0,0593
0,9	-1,3726	-0,6044	-0,6954	-0,7751
1,0	-1,2940	-1,1442	-1,2208	-1,2718

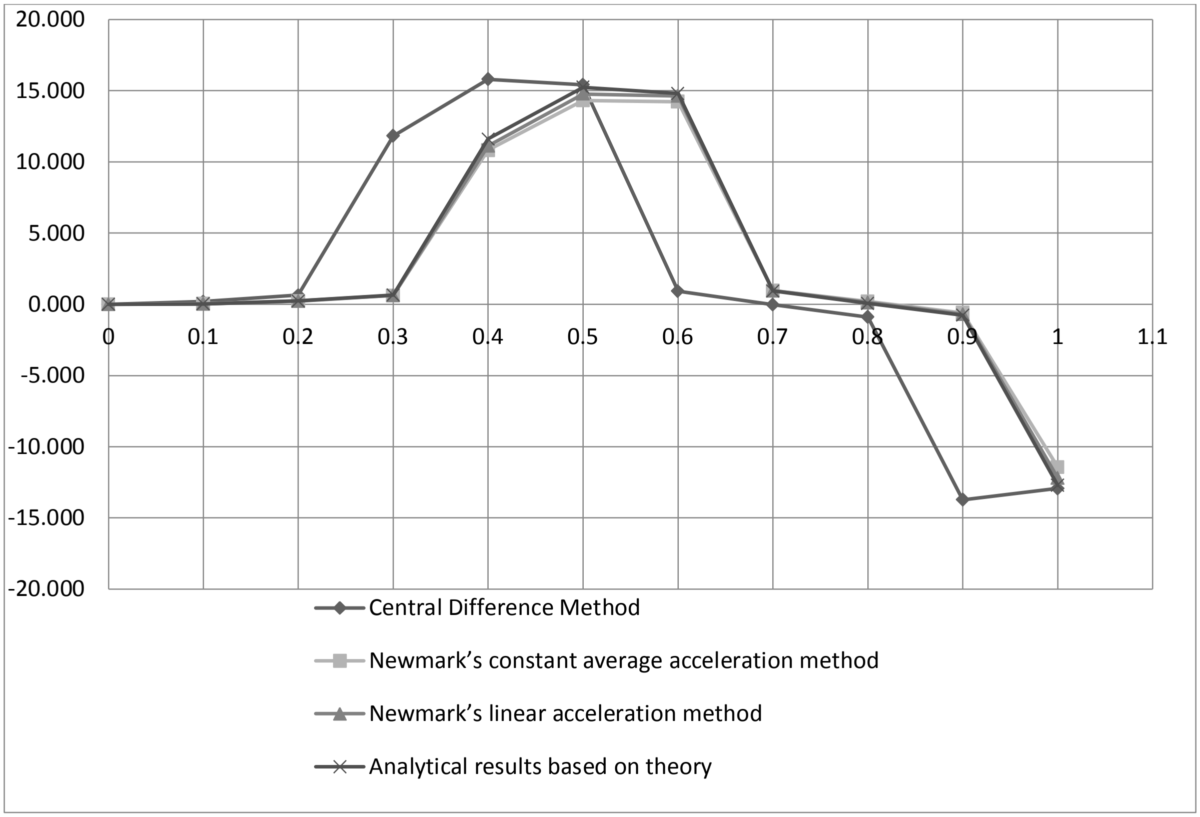

Figure 3: Compare results of displacement values between 3 methods.

4 CONCLUSION

This study deals with three methods of calculating dynamic responses of a single-degree of freedom oscillator, i.e., central difference method (CDM) and Newmark's method (NBM), finite element method (FEM) using recorded ground acceleration for 60seconds.

The study shows that by comparing all the methods for the SDOF linear problem with the time step of 0.1 sec the Newmark's average acceleration method is the most accurate method from the other three methods which are being compared as it gives almost similar results to the theoretical one. Therefore, the study concludes that to get optimum accuracy we can use average acceleration method when the time step is 0.1 sec.

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NGHIÊN CỨU VÀ SO SÁNH CÁC PHƯƠNG PHÁP SỐ TRONG PHÂN TÍCH BÀI TOÁN ĐỘNG LỰC HỌC HỆ MỘT BẬC TỰ DO

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Tóm tắt. Ngày nay với việc phát triển ngày càng mạnh mẽ của khoa học máy tính. Việc ứng dụng phương pháp phần tử hữu hạn để giải các bài toán kỹ thuật ngày càng trở nên phổ biến. Tuy nhiên, việc giải và phân tích số chỉ cho kết quả gần đúng với lời giải giải tích. Hiện có nhiều phương pháp số cho các kỹ sư lựa chọn, mỗi phương pháp có những ưu và nhược điểm riêng và mức độ chính xác cũng khác nhau. Trong bài báo này, tác giả so sánh kết quả của ba phương pháp: Phương pháp sai phân trung tâm, phương pháp gia tốc trung bình không đổi, phương pháp gia tốc tuyến tính với lời giải lý thuyết chính xác. Thông qua kết quả thu được, các kỹ sư sẽ có được cái nhìn tổng quát hơn về các phương pháp và lựa chọn cho mình một công cụ tính toán hợp lý để giải quyết các vấn đề tương tự.

Từ khóa. Phương pháp số, phương pháp phần tử hữu hạn, phương pháp sai phân trung tâm, phương pháp gia tốc trung bình không đổi, phương pháp gia tốc tuyến tính.

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