ON COFINITELY LIFTING MODULES

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Abstract. In this paper we define cofinitely lifting module as generalizations of lifting and supplemented modules. In this paper, new characterizations of these modules are obtained and several properties of this module are proved.
Keywords: δ-small modules, δ-lifting modules, amply δ-supplemented modules.

1. INTRODUCTION

Throughout this note, R is an associative ring with unit and all modules are unital right R-modules.
We review some basic definitions. A submodule N of a module M is called *small*, written $N \ll M$, if $M \neq N + L$ for every proper submodule L of M. A module M is called *lifting* if, for all $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in B (Keskin, 1998; Keskin, 2000; Mohamed & Muller, 1990 and Wisbauer, 1991). A submodule N of M is called *cofinite* (in M) if M/N is a finitely generated module. Following (Alizade et al., 2001), a module M is called *cofinitely supplemented* if every cofinite submodule of M has a supplemented. Recall that a R-module M is called $\oplus$-*cofinitely supplemented* if every cofinite submodule of M has a supplemented that is a direct summand of M. A submodule N of M has ample supplements in M if every submodule L such that $M = N + L$ contains a supplement of N in M. The module M is called *amply cofinitely supplemented* if every cofinite submodule of M has ample supplements in M. For any unexplained terminology please refer (Anderson & Fuller, 1974; Goodearl, 1976; Tütüncū, 2005 and Wang & Ding, 2006).

2. ON COFINITELY LIFTING MODULES

Definition 2.1. *The module M is called cofinitely lifting if, for every cofinite submodule N of M, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is small in B.*
We begin the following lemma:
Lemma 2.2. *The following conditions are equivalent for a module M :*
(1) M is cofinitely lifting.
(2) Every cofinite submodule $U \leq M$ has a supplement V in M such that $U \cap V$ is a direct summand of U.
(3) For all cofinite submodule N of M, there exists a decomposition $N = A \oplus B$ such that A is a direct summand of M and $B \ll M$.
(4) For all cofinite submodule N of M, there exists a direct summand A of M such that $A \leq N$ and $N/A \ll M/A$.
(5) For every cofinite submodule $U \leq M$, there exists an idempotent $e \in End(M)$ with $e(M) \leq U$ and $(1-e)(U) \ll (1-e)(M)$.
Proof. (1) ⇒ (2) Let U be a cofinite submodule of M. Then, there exists a decomposition $M = A \oplus V$ such that $A \leq U$ and $U \cap V$ is small in V. It follows that $M = U + V$ and $U = A \oplus U \cap V$, and so V is a supplement of U in M.
(2) ⇒ (3) Let N be a cofinite submodule of M. By (2), there is N' a supplement of N in M such that $N \cap N'$ is a direct summand of N. Call $B = N \cap N'$. Then, we have $N = A \oplus B$ for some submodule A of N. It follows that $M = A \oplus N'$ and $B \ll M$.

(3) ⇒ (4) Let N be a cofinite submodule of M. By (3), we have $N = A \oplus B$ and $M = A \oplus N'$ with $B \ll M$. One can check that $N/A \ll M/A$.
(4) ⇒ (5) Let U be a cofinite submodule of M. By (4), there exists an idempotent $e \in \mathrm{End}(M)$ with $e(M) \leq U$ and $U/e(M) \ll M/e(M)$. Then, we have $(e(M) + [U \cap (1-e)M]) \ll M/e(M)$.

Note that $(1-e)(U) \cong (e(M) + [U \cap (1-e)M])$ and $(1-e)(M) \cong M/e(M)$ and so $(1-e)(U) \ll (1-e)(M)$.
(5) ⇒ (1) Let N be a cofinite submodule of M. By (5), there exists an idempotent $e \in \mathrm{End}(M)$ with $e(M) \leq N$ and $(1-e)(N) \ll (1-e)(M)$. It follows that $M = e(M) \oplus (1-e)(M)$ and

$$N \cap (1-e)(M) = (1-e)(N) \ll (1-e)(M).$$

We deduce that M is cofinitely lifting.
Corollary 2.3. *Every indecomposable cofinitely lifting module is hollow.*
Corollary 2.4. *A noetherian right R-module M is a cofinitely lifting module if and only if M is lifting.*
By the definition of cofinitely lifting modules, we have the following lemma.
Lemma 2.5. *Every cofinite direct summand of a cofinitely lifting module is cofinitely lifting.*

Next, we give sufficient conditions for a factor module of a cofinitely lifting module to be cofinitely lifting and for a direct sum of two cofinitely lifting modules to be cofinitely lifting. A submodule X of a module M is called *fully invariant* if for every $h \in \mathrm{End}(M_R), h(X) \leq X$. A module M is called *distributive* if its lattice of submodules is a distributive lattice, that is, $A \cap (B + C) = (A \cap B) + (A \cap C)$ for any submodules A, B and C of M.
Proposition 2.6. *Let M be a cofinitely lifting module and $X \leq M$. Then M/X is cofinitely lifting in each of the following cases:*
(1) For every direct summand K of $M, (K + X)/X$ is also a direct summand of M/X.
(2) M is a distributive module.
(3) For any $e^2 = e \in End(M)$, $eX \leq X$. In particular, X is a fully invariant submodule of M.
(4) K is a direct summand and cofinite.
Proof. (1) Let A/X be a cofinite submodule of M/X. Then A is a cofinite submodule of M. Since M is a cofinitely lifting module, there exists a direct summand K of M such that $K \leq A$ and A/K is small in M/K by Lemma 2.2. By hypothesis, $(K + X)/X$ is also a direct summand of M/X. Clearly, $(K + X)/X \leq A/X$. Now $A/(K + X)$ is small in $M/(K + X)$. Hence, M/X is cofinitely lifting.
(2) Let $M = K \oplus L$. Then $M/X = ((K + X)/X) + ((L + X)/X)$ and $X = X + (K \cap L) = (X + K) \cap (X + L)$. So $M/X = ((K + X)/X) \oplus ((L + X)/X)$. By (a), M/X is cofinitely lifting.
(3) Let $M = K \oplus L$. Consider the projection map e of M into K with kernel $(1-e)M = L$. Then $e^2 = e \in \mathrm{End}(M)$ and $eM = K$. By hypothesis, $eX \leq X$ and and $(1-e)X \leq X$. Hence, $eX = X \cap K$ and $(1-e)X = X \cap L$. Therefore, $X = (X \cap K) \oplus (X \cap L)$. Now $(K + X)/X = (K \oplus (XL)) = X$ and $(L + X)/X = (L \oplus (X \cap K))/X$. Hence, $M = K + X + L + X = (K \oplus (X \cap L)) + L + X$ implies that $M/X = (K \oplus (X \cap L))/X + (L + X)/X$. Since $(K \oplus (X \cap L)) \cap (L + X) = (X \cap L) \oplus (X \cap K) = X, M/X = (K \oplus (X \cap L))/X \oplus (L + X)/X$. Thus, by (a) M/X is cofinitely lifting.
Corollary 2.7. *Let M be cofinitely supplemented module. Then $M/Rad(M)$ is cofinitely lifting.*
Theorem 2.8. *Let $M = M_1 \oplus M_2$. If M_1 and M_2 are cofinitely lifting modules such that M_1 is M_2-projective, then M is a cofinitely lifting module.*
Proof. Let N be a cofinite submodule M. We have

$$M_1/[M_1 \cap (M_2 + N)] \cong (M_1 + M_2 + N)/(M_2 + N) \cong M/(M_2 + N),$$

so that $M_1 \cap (M_2 + N)$ is a cofinite submodule of M_1. Since M_1 is a cofinitely lifting modules, there exists $K \leq M_1 \cap (N + M_2)$ such that $M_1 = K \oplus K'$ and $K' \cap (N + M_2) \ll M_1$. Therefore

$$M = K \oplus K' \oplus M_2 = N + (K' \oplus M_2)$$

Since M_1 is M_2-projective, K is $K' \oplus M_2$-projective. By (Koçan, 2007, Lemma 2.6), there exists a submodule N_1 of N such that $M = N_1 \oplus (K' \oplus M_2)$. Then $N \cap (L + K') = L \cap (N + K')$ for any submodule L of M_2. On the other hand, M_2 is cofinitely lifting, there is a submodule X of $M_2 \cap (N + K') = N \cap (M_2 \oplus K')$ such that $M_2 = X \oplus Y$ and $Y \cap (N + K') \ll M_2$ for some $Y \leq M_2$. Hence $M = (N_1 \oplus X) \oplus (Y \oplus K')$. We have $N_1 \oplus X \leq N$ and $N \cap (Y \oplus K') = Y \cap (N + K')$. But $Y \cap (N + K') \ll Y$. Then $N \cap (Y \oplus K') \ll Y \oplus K'$. Thus M is a cofinitely lifting module.

Corollary 2.9. *Let $M = M_1 \oplus M_2$ be a projective module. If M_1 and M_2 are cofinitely lifting modules, then M is a cofinitely lifting module*.

A module M is called *duo*, if every submodule of M is fully invariant.

Proposition 2.10. *Let $M = M_1 \oplus M_2$ be a duo module. If M_1 and M_2 are cofinitely lifting modules, then M is cofinitely lifting.*

Proof. Assume M_1 and M_2 are cofinitely lifting modules. Take any cofinite submodule L of M. Then $L = (L \cap M_1) \oplus (L \cap M_2)$. Clearly, $L \cap M_1$ and $L \cap M_2$ are cofinite submodules of M_1 and M_2, respectively. For each i, there exists some direct summands D_i of M_i such that $M_i = D_i \oplus D_i'$ with $L \cap M_i \leq D_i$ and $L \cap D_i' \ll D_i'$. Therefore $M = (D_1 \oplus D_1') \oplus (D_2 \oplus D_2') = (D_1 \oplus D_2) \oplus (D_1' \oplus D_2')$. We have $L \leq D_1 \oplus D_2$ and $(D_1 \oplus D_2) \cap (D_1' \oplus D_2') \ll D_1' \oplus D_2'$.

Corollary 2.11. *Let $M = M_1 \oplus M_2$ be a module with $R = r_R(m_1) + r_R(m_2)$ for all $m_1 \in M_1$ and $m_2 \in M_2$. If M_1 and M_2 are cofinitely lifting modules, then M is cofinitely lifting.*

Proof. If R satisfies the condition $R = r_R(m_1) + r_R(m_2)$ for all $m_1 \in M_1$ and $m_2 \in M_2$, then $L = (L \cap M_1) \oplus (L \cap M_2)$ for all submodules L of M. By the proof of Proposition 2.10 , we have that M is cofinitely lifting.

Of course, every cofinitely lifting module is $\oplus$-cofinitely supplemented. But the converse in general is not true.

Example 2.12. Let $R = \mathbb{Z}_8$. Then R_R is perfect and so R_R is a cofinitely lifting module. We have $2R/4R$ is a simple R-module, $2R/4R$ is cofinitely lifting. Let $M = R \oplus (2R/4R)$, then M is a finitely generated R-module. If M_R is cofinitely lifting, then M is lifting. This is a contradiction by (Koçan, 2007, Example 2.4) and so M_R is not cofinitely lifting. Moreover, M is a $\oplus$-cofinitely supplemented module by (Calişici & Pancar, 2004, Theorem 2.6).

Following (Wisbauer, 1991, page 359], a module M is called *π-projective* if whenever N and L are submodules of M with $M = N + L$, there exists an endomorphism α of M such that $\alpha(M) \leq N$ and $(1-\alpha)(M) \leq L$. As Wisbauer points out, π-projective supplemented modules are amply supplemented (Wisbauer, 1991, 41.15]).

The following proposition indicate that a $\oplus$-cofinitely supplemented and projective module is cofinitely lifting.

Proposition 2.13. *Let M be a projective module. Then M is cofinitely lifting if and only if M is $\oplus$-cofinitely supplemented.*

Proof. $(\Rightarrow)$ is clear.

$(\Leftarrow)$. If M is $\oplus$-cofinitely supplemented, then R is cof-semiperfect by (Calişici & Pancar, 2005, Theorem 2.1) and so for every finitely generated factor module of M has a projective cover. Thus M is cofinitely lifting.

From this above Proposition, we have the following example:

Example 2.14. (Calişici & Pancar, 2005) Let R denote the ring $K[[x]]$ of all power series $\sum_{i=0}^{\infty} k_i x^i$ in an indeterminate x and with coefficients from a field K which is a local ring. Then $R^{(N)})$ is $\oplus$-cofinitely supplemented and so $R^{(N)}$ is cofinitely lifting by Proposition 2.13. But $R^{(N)}$ is not lifting.

We have

$$\text{lifting} \overset{\not\Leftarrow}{\Rightarrow} \text{cofinitely lifting} \overset{\not\Leftarrow}{\Rightarrow} \oplus\text{-cofinitely supplemented.}$$

It is easy to prove that the following proposition.

Proposition 2.15. *Let R be a ring. Then the following are equivalent:*

(1) R is semiperfect.

(2) Every free module is cofinitely lifting.

Corollary 2.16. *Let R be a ring. Then the following are equivalent:*

(1) R is semiperfect.

(2) Every projective module is cofinitely lifting.

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MÔĐUN NÂNG ĐỐI HỮU HẠN

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Abstract. Trong bài báo này chúng tôi định nghĩa môđun nâng đối hữu hạn như một mở rộng của môđun phần bù và môđun nâng. Cũng trong bài báo, các đặc trưng mới của các môđun này được đưa ra và một số tính chất cũng đã được chúng tôi chứng minh.
Keywords: môđun δ-bé, môđun δ-nâng, môđun δ-phần bù đầy đủ.

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