Journal of Science and Technology, Vol. 60, 2022

MODULES AND RINGS IN WHICH EVERY COMPLEMENT IS ISOMORPHIC TO A SUMMAND

NGUYEN THI THU HA
Faculty of Fundamental Science, Industrial University of Ho Chi Minh City
nguyenthithuha@iuh.edu.vn
DOIs: https://doi.org/10.46242/jstiuh.v60i06.4634

Abstract. In this paper, we introduce a generalization of the well-known CS condition. We say an R-module M is a $CIS\text{-}module$ if every complement is isomorphic to a summand. We prove that if R is a right CIS-ring right FGF ring, then R is a quasi-Frobenius ring, and if R is a right CIS-ring right CF ring, then R is a right artinian ring. New characterizations of quasi-Frobenius rings are provided by using CIS-rings. Moreover, many of the important propositions related to CS-rings are generalized to CIS-rings also presented.
Keywords. CIS-module, CIS-ring, quasi-Frobenius ring, CF ring, FGF ring.

1. INTRODUCTION

Throughout this paper, R is a ring with unity and we use M_R (${}_RM$) to denote a unital right (left) R-module. A submodule C is called a *closed (or complement) submodule* of M if it has no proper essential extensions in M. A module M is CS (or extending) if every submodule of M is essential in a direct summand of M; or equivalently, every complement is a summand. CS-modules have an important place in module and ring theory (please see Dung et al., 1994). Many authors studied on CS-modules and its generalizations (Tercan & Yucel, 2015). In this paper, we say a module M is a $CIS\text{-}module$ if every complement is isomorphic to a summand. Firstly we remark that, clearly, CIS-modules is a real generalization of CS modules. Example 2.2 demonstrates that there are some CIS-modules which are not CS. A necessary condition for the equivalence of CS and GIS conditions are given in Proposition 2.3 that if a module M is C2, then M is CS if and only if M is CIS. From properties of CIS-modules, we have that an R-module M is a CIS-module if and only if every submodule of M is essential in a complement submodule of M which is isomorphic to a direct summand of M; or equivalently, for submodules A and B of M with $A \cap B = 0$, there exists a complement submodule C of M such that $B \leq C$, $A \cap C = 0$ and C is isomorphic to a direct summand of M.

In this paper, we consider the relationship between CIS-rings and some other well-studied classes of rings, and in the articles that (Pardro & Asensio, 1997; Yousif, 1997; Nicholson & Yousif, 1998), we generalized all the results obtained with CS-rings to CIS-rings. Gómez Pardo and Guil Asensio proved in (Pardro & Asensio, 1997) that if a ring R is a right CS-ring, then every direct summand of injective envelope $E(R_R)$ has an essential finitely generated projective submodule. We generalize this proposition in Proposition 2.5 by proving that if R is a right CIS-ring, then every direct summand of injective envelope $E(R_R)$ has an essential finitely generated projective submodule. As results of Proposition 2.5, we proved in Corollary 2.8 and 2.9 that FGF and CF conjectures are true when the ring is a CIS-ring. Recall that a ring R is *quasi-Frobenius* if and only if each right R-module embeds in a projective module; or equivalently, embeds in a free module. A ring R is called a *right FGF ring* if each finitely generated right R-module embeds in a free module. A ring R is called a *right CF ring* if each cyclic right R-module embeds in a free module. Obviously, every right FGF ring is a right CF ring. The converse of this proposition is not true, in general. Björk provided in (Björk, 1969) an example which illustrates that there is a left CF ring which is not a left FGF ring. Let us recall CF and FGF conjectures:

1. **The FGF conjecture:** Should a right FGF ring be a QF ring?
2. **The CF conjecture:** Should a right CF ring be a right artinian ring?

Björk (Björk, 1972) and Tolskaya (Tolskaya, 1970) unaware of each other proved that every right selfinjective right CF ring is right artinian. In (Pardro & Asensio, 1997), Gómez Pardo and Guil Asensio generalize their theorem by proving that every right CS right CF ring is right artinian (please see Jain et al., 2012, p.113). Now, we extend their result in Corollary 2.8 by proving that every right CIS right CF ring is right artinian. In (Pardro & Asensio, 1997), Gómez Pardo and Guil Asensio also proved that if R is a right CS right FGF ring, then R is QF. We prove in Corollary 2.9 that if R is a right CIS right FGF ring, then R is QF. Then, we prove in Theorem 2.10 that a ring R is a right Kasch and a left CIS-ring if and only if R is semiperfect left continuous with $Soc(R_R) \leq^e {}_RR$. In Proposition 2.12, we prove that if R is a left CIS-ring such that every cyclic right R-module is torsionless, then R is a semiperfect, left continuous ring with $soc(R_R) \leq^e {}_RR$. In particular, R is left finite dimensional. In Proposition 2.13, we prove that let R be a right cogenerator ring: (1) If R is a left CIS-ring then R is a left continuous and a right selfinjective ring (and so is a right PF-ring). (2) If $R \oplus R$ is CIS as a left R-module then R is left and right PF. In Thorem 2.14, we show that let R be a left CIS-ring R. Then, R is a right PF-ring if and only if $J(R) \subseteq Z(R_R)$ and every 2-generated right R-module is torsionless. In Theorem 2.15, we prove that let R be a left CIS-ring such that every cyclic right R-module embeds in a free module (i.e., right CF ring). Then, R is QF if and only if $J(R) \subseteq Z(R_R)$; or equivalently, $Soc(R_R) \subseteq Soc({}_RR)$; or equivalently, R is right mininjective. In Theorem 2.19, we prove that a ring R is QF if and only if R is a right Johns and a left CIS-ring; or equivalently, R is left P-injective, left CIS-ring and right Noetherian ring; or equivalently, R is left P-injective, left CIS-ring with ACC on left annihilators; or equivalently, R is left P-injective, left CIS ring and satisfy ACC on right annihilators; or equivalently, R is a right CIS-ring and every 2-generated right R-module embeds in a free module; or equivalently, R is a left GP-injective, left CIS-ring with ACC on essential left ideals. In Theorem 2.20, we prove that a ring R is a right Johns, left Kasch ring if and only if R is a right CIS-ring and a right CF ring. In Theorem 2.21 we prove that a ring R is a right CIS-ring which is left and right Kasch; if and only if R is right Kasch and a right continuous ring; or eqivalently, R is a semiperfect right continuous ring with essential right socle. We prove in Theorem 2.22 that a ring R is a left and right Kasch left and right CIS-ring if and only if R is a left and right CIS-ring, and the dual of every simple right R-module is simple; or equivalently, R is semiperfect left and right continuous ring with $Soc({}_RR) = Soc(R_R)$ essential as a left and as a right R-module in R. We show in Theorem 2.25 a ring R has a perfect duality if and only if or equivalently, R is left and right Kasch and $R \oplus R$ is a left and right CIS-module; or equivalently, the dual of every simple right R-module is simple and $R \oplus R$ is a left and right CIS-module; or equivalently, the dual of every simple right H-module is simple and H is a right and left CIS-ring, where $H = M_2(R)$. At the end of the paper, we prove in Corollary 2.27 a ring R is quasi-Frobenius if and only if R is right and left Kasch and a right countably Σ-CIS-ring.

Recall from (Smith, 1992) that a module M is a *UC-module* if every submodule has a unique closure; or equivalently, the intersection of every pair of complement submodules of M is again a complement submodule of M. A module M is a *C2-module* if a submodule A of M is isomorphic to a direct summand of M, then A is the direct summand of M. A module M is called a *continuous module* if M is both a CS and a C2-module. A ring R is called a right *C2-ring* (*continuous ring*) if the module R_R is a C2-module (continuous module). Left C2-rings can be defined similarly.

The left (right) annihilator of a subset I of a ring R is denoted $l(X)$ (r(X)). $M^{(n)}$ denotes the direct sum of n copies of M. By $\mathbb{Z}$ and $\mathbb{R}$, we denote the ring of integer and real numbers, respectively. $\mathbb{Z}_n$ will denote $\mathbb{Z}/n\mathbb{Z}$. $M_n(R)$ and $Soc(M)$ denote the $n \times n$ matrix ring over R and the socle of a module M, respectively. For any unexplained terminology please refer (Dung et al., 1994; Nicholson & Yousif, 2003).

2. CIS-MODULES AND RINGS

Definition 2.1. *A module M is called a CIS-$module$ if every complement submodule of M is isomorphic to a direct summand of M. A ring R is called a right CIS-$ring$ if the module R_R is a CIS-module. Left CIS-rings can be defined similarly. A ring R is called a CIS-ring if it is left and right CIS.*

Clearly, uniform, semisimple and CS-modules are CIS-modules. Recall from (Behboodi et al., 2018) that a module M is a *virtually semisimple module* if every submodule is isomorphic to a direct summand. Clearly, every virtually semisimple module is a CIS-module.

Example 2.2. *There exist some CIS-modules which are not CS:*

1. Let A be a commutative principal ideal domain (PID) and $R = \begin{bmatrix} A & A \\ 0 & A \end{bmatrix}$ the 2×2 generalized triangular matrix ring. R_R is a CIS-module but not a CS-module.

2. Let $R = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{bmatrix}$ be the 2×2 generalized triangular matrix ring. R_R is a not a CS-module by (Tercan & Yucel, 2015, Example 5.102) but it is CIS.

Proposition 2.3. *If a module M is a C2-module, then M is CS if and only if M is CIS.*

Proof. The necessity is clear. The other direction follows immediately from the definition of the C2-module.

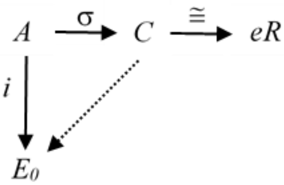
Corollary 2.4. *A module M is a continuous module if and only if M is both a CIS and a C2-module.*

In this section, we investigate the relationships between CIS-rings and some other important classes of rings.

Gómez Pardo and Guil Asensio proved in (Padro & Asensio, 1997, Theorem 2.5) that, let R be a ring and P_R a finitely generated projective module such that each direct summand of injective envelope $E = E(P_R)$ has an essential finitely generated projective submodule and $|\Omega(R)| \leq |C(P)|$. Then P_R cogenerates the simple right R-modules and has finite essential socle, where $\Omega(R)$ denotes a set of representatives of the isomorphism classes of simple right R-modules, and for a right R-module M, $C(M)$ denotes a set of representatives of the isomorphism classes of simple submodules of M, and $|X|$ denotes the cardinality of a set X. They also proved that if a ring R is a right CS-ring, then every direct summand of injective envelope $E(R_R)$ has an essential finitely generated projective submodule. We begin this section by generalizing this proposition:

Proposition 2.5. *If R is a right CIS-ring, then every direct summand of injective envelope $E(R_R)$ has an essential finitely generated projective submodule.*

Proof. Let R be a right CIS-ring and $\varepsilon: R_R \hookrightarrow E$ the injective envelope of R_R. Let E_0 be a direct summand of E. Then $A = R \cap E_0$ is essential in E_0. Moreover, Since R_R is CIS, A is essentially embeds in a complement submodule C of M such that C is isomorphic to a direct summand eR of R_R. Call $\sigma: A \to C$ the inclusion. By the injectivity, σ extends to a homomorphism $\theta: C \to E_0$.



Then θ is monomorphism since A is essential in C and $\theta|_A = i$. Now, we have $Im(\theta) \cong C \cong eR$. Thus, $Im(\theta)$ is a finitely generated projective (indeed, cyclic) submodule of E_0, which is essential since it contains A.

A ring R is called *right cogenerator* if every right R-module is torsionless, and R is called a *right PF-ring* (*pseudo-Frobenius ring*) if it is right cogenerator and right selfinjective; or equivalently, it is semiperfect, right selfinjective and $Soc(R_R) \leq^e R_R$. Osofsky proved in (Osofsky, 1996) that a right injective cogenerator ring R (i.e., a right PF-ring) has finite essential right socle and is, therefore, semiperfect. In (Padro & Asensio, 1997, Corollary 2.7), Gómez Pardo and Guil Asensio generalized Osofsky's result and proved that if a ring R is CS and cogenerates the simple right R-modules, then R_R has finite essential socle. From Proposition 2.5, we obtain the following corollary which generalizes (Padro & Asensio, 1997, Corollary 2.7):

Corollary 2.6. *Let R be a ring such that R_R is CIS and cogenerates the simple right R-modules. Then R_R has finite essential socle.*

In (Padro & Asensio, 1997, Corollary 2.8), the authors proved that a ring R is right PF if and only if R_R is a CS cogenerator; or equivalently, R_R is a cogenerator, and every direct summand of injective envelope $E(R_R)$ contains an essential finitely generated projective submodule. We obtain in Corollary 2.7, which extends (Padro & Asensio, 1997, Corollary 2.8), a characterization of the right PF-rings by strengthening the hypothesis of the previous corollary:

Corollary 2.7. *The following conditions are equivalent for a ring R:*

1. *R is right PF;*
2. *R_R is a CIS cogenerator;*
3. *R_R is a cogenerator, and every direct summand of injective envelope $E(R_R)$ contains an essential finitely generated projective submodule.*

Proof. (1) ⇒ (2) Since R is a right PF-ring then R_R is a CS cogenerator by (Padro & Asensio, 1997, Corollary 2.8). So, it is CIS.

(2) $\Rightarrow$ (3) Clear.

(1) $\Leftrightarrow$ (3) is proved in (Padro & Asensio, 1997, Corollary 2.8).

Recall that if R is a ring such that each cyclic right R-module has finite essential socle, then it is right artinian. In (Padro & Asensio, 1997, Corollary 2.9), the authors proved that if R is a right CS-ring such that every cyclic right R-module embeds in a free module, then R is right artinian. By Corollary 2.7, we have the following corollary which generalizes (Padro & Asensio, 1997, Corollary 2.9):

Corollary 2.8. *Let R be a right CIS-ring and every cyclic right R-module embeds in a free module (i.e, right CF ring). Then R is right artinian.*

In (Padro & Asensio, 1997, Corollary 2.10), the authors proved that if R is a right FGF ring and every direct summand of the injective envelope $E(R_R)$ contains an essential finitely generated projective submodule, then R is QF. Now, we can state the following corollary:

Corollary 2.9. *If R is a right CIS right FGF ring, then R is QF.*

A ring R is called *right Kasch* if every simple right R-module embeds in R; or equivalently, $l(A) \neq 0$ for every maximal right ideal A of R. Nicholson and Yousif proved in (Nicholson & Yousif, 2003, Theorem 4.10) that R is a right Kasch and a left CS-ring if and only if R is semiperfect left continuous with $soc(R_R) \leq^e {}_R R$. The next theorem extends (Nicholson & Yousif, 1998, Lemma 2.3) and (Nicholson & Yousif, 2003, Theorem 4.10).

Theorem 2.10. *A ring R is a right Kasch and a left CIS-ring if and only if R is semiperfect left continuous with $Soc(R_R) \leq^e {}_R R$.*

Proof. ($\Rightarrow$:) Every right Kasch ring is a left C2-ring by (Yousif, 1997, Lemma 1.15). Then, by Proposition 2.3, R is a left CS-ring. The rest is follows by (Nicholson & Yousif, 2003, Theorem 4.10).

($\Leftarrow$:) Clear by (Nicholson & Yousif, 2003, Theorem 4.10).

The next corollary generalizes (Nicholson & Yousif, 2003, Corollary 4.13).

Corollary 2.11. *The following conditions are equivalent for a ring R:*

1. *R is a left CIS, left and right Kasch ring;*
2. *R is a semiperfect left continuous ring with essential left socle.*

Proof. (1) $\Rightarrow$ (2) R is semiperfect and left continuous by Theorem 2.10. Since R is also left Kasch, it follows from (Nicholson & Yousif, 2003, Lemma 4.5) that $Soc({}_R R) \leq^e {}_R R$.

(2) $\Rightarrow$ (1) It follows from (Nicholson & Yousif, 2003, Lemma 4.11(4)).

A right R-module M is called *torsionless* if M is embedded in a direct product of copies of R. Recall that if A is a right ideal of R, then R/A is torsionless as a right R-module; or equivalently, $rl(A) = A$. In (Nicholson & Yousif, 1998, Proposition 2.4), the Nicholson and Yousif showed that if R be a left CS-ring such that every cyclic right R-module is torsionless, then R is a semiperfect, left continuous ring with $soc(R_R) \leq^e {}_R R$. In particular, R is left finite dimensional. The next proposition generalizes (Nicholson & Yousif, 1998, Proposition 2.4).

Proposition 2.12. *Let R be a left CIS-ring such that every cyclic right R-module is torsionless. Then R is a semiperfect, left continuous ring with $soc(R_R) \leq^e {}_R R$. In particular, R is left finite dimensional.*

Proof. We have $rl(A) = A$ for every right ideal A because R/A is torsionless. In particular, R is right Kasch and so is left continuous with $soc(R_R) \leq^e {}_R R$ by Theorem 2.10. Furthermore, since R is left CS, then every complement left ideal is a summand, and so is principal. Thus, R is semiperfect by (Nicholson & Yousif, 1998, Lemma 2.2), so write $R = Re_1 \oplus \ldots \oplus Re_n$, where each e_i, is a local idempotent. Then each Re_i is a CS-module and so is uniform. So, R is left finite dimensional.

Gómez Pardo and Guil Asensio proved in (Pedro & Asensio, 1997) that, if R be a right cogenerator ring, right CS then it is right selfinjective, or in other words, R is a right PF-ring (i.e., R is right cogenerator, right selfinjective) if and only if R is a right cogenerator, right CS-ring. Then, we proved in Corollary 2.7 that R is a right PF-ring if and only if R is a right cogenerator, right CIS-ring. Hereby, Corollary 2.7 extends all the known results on the subject. On the other hand, recall from (Faith, 1976) that a ring R is left and right PF if and only if R is right cogenerator, left selfinjective. Therefore, it is natural to ask that whether the result of Gómez Pardo and Guil Asensio can be obtained if the right CS-condition by the left CS-condition are replaced. Nicholson and Yousif gave an affirmative answer to this question in ((Nicholson & Yousif, 1998, Proposition 2.5), and now we extend Nicholson and Yousif's result in the following proposition:

Proposition 2.13. *Let R be a right cogenerator ring, then:*

1. *If R is a left CIS-ring then R is a left continuous and a right selfinjective ring (and so is a right PF-ring).*
2. *If $R \oplus R$ is CIS as a left R-module then R is left and right PF.*

Proof. (1) R is a semiperfect, left continuous ring by Proposition 2.12. In particular R has a finite number of isomorphism classes of simple right (and left) R-modules. Since R is a right cogenerator ring, R is a right selfinjective ring by (Faith, 1976, Proposition 24.9). So, R is right PF.

(2) R is a left continuous ring by (1). Then it is a left selfinjective ring by (Yousif, 1997, Proposition 1.21). Thus R is left PF, and hence it is right PF by (1).

In (Nicholson & Yousif, 1998, Theorem 2.8), the Nicholson and Yousif proved that let R be a left CS-ring. Then R is a right PF-ring if and only if $J(R) \subseteq Z(R_R)$ and every 2-generated right R-module is torsionless. The next theorem extends (Nicholson & Yousif, 1998, Theorem 2.8).

Theorem 2.14. *The following conditions are equivalent for a left CIS-ring R:*

1. *R is a right PF-ring;*
2. *$J(R) \subseteq Z(R_R)$ and every 2-generated right R-module is torsionless.*

Proof. $(1) \Rightarrow (2)$ is proved in [3, Theorem 2.8].

$(2) \Rightarrow (1)$ By Proposition 2.12, R is a semiperfect, left continuous ring with $Soc(R_R) \leq^e {}_RR$. So, R is a left CS-ring. Now, the proof is clear by (Nicholson & Yousif, 1998, Theorem 2.8).

Recall from (Anderson & Fuller, 1974) that a right artinian ring R is QF if and only if $Soc({}_RR) = Soc(R_R)$ and $Soc(eR)$ and $Soc(Re)$ are simple for every local idempotent e of R. Recall that we proved Corollary 2.8 in that if R is a right CIS-ring and every cyclic right R-module embeds in a free module, then R is right artinian. A ring R is called *right mininjective* if each R-homomorphism from a simple right ideal to R is given left multiplication (Nicholson & Yousif, 1997).

Theorem 2.15. *Let R be a left CIS-ring such that every cyclic right R-module embeds in a free module (i.e., right CF ring). The following conditions are equivalent:*

1. *R is QF;*
2. *$J(R) \subseteq Z(R_R)$;*
3. *$Soc(R_R) \subseteq Soc({}_RR)$;*
4. *R is right mininjective.*

Proof. $(1) \Rightarrow (2)$ Clear because R is right selfinjective.

$(2) \Rightarrow (3)$ R is semiperfect by Proposition 2.12, and hence $r(J) = Soc({}_RR)$. From (2) $Soc(R_R) \subseteq r[Z(R_R)] \subseteq r(J) = Soc({}_RR)$.

$(3) \Rightarrow (1)$ It can easily be see that $Soc(R_R) \leq^e {}_RR$ by Proposition 2.12. Therefore, $Soc({}_RR) \subseteq Soc(R_R)$. So, $Soc({}_RR) = Soc(R_R)$ by (3). Now, R is semiperfect by Proposition 2.12, Thus, we write $R = Re_1 \oplus \ldots \oplus Re_n$ where $\{e_1, \ldots, e_n\}$ is a complete set of local orthogonal idempotents. Moreover R is left continuous by Proposition 2.12. Then, R is a left CS-ring. The rest follows by (Nicholson & Yousif, 1998, Theorem 2.9).

$(4) \Rightarrow (3)$ It is immediate by (Nicholson & Yousif, 1997, Theorem 1.14).

$(3) \Rightarrow (4)$ It is obvious because of $(3) \Rightarrow (1)$.

Corollary 2.16. *[Nicholson & Yousif, 1998, Theorem 2.9] If R is a left CS-ring such that every cyclic right R-module embeds in a free module then the conditions from (1) to (4) in Theorem 2.15 are equivalent.*

A ring R is called a *right weakly continuous ring* if R is semiregular and $J = Z_r$. Recall from (Nicholson & Yousif, 2003, Theorem 1.35) that R is right self-injective if and only if $R \oplus R$ is continuous (equivalently quasi-continuous) as a right R-module. The following theorem generalizes (Nicholson & Yousif, 2003, Corollary 7.41).

Theorem 2.17. *The following are equivalent for a ring R:*

1. *R is right self-injective;*
2. *$(R \oplus R)_R$ is a CIS-module and a C2-module;*
3. *R is right weakly continuous and $(R \oplus R)_R$ is a CIS-module.*

Proof. $(1) \Rightarrow (3)$ clear by (Nicholson & Yousif, 2003, Corollary 7.41).

$(3) \Rightarrow (2)$ If R is right weakly continuous, so also is $M_2(R) \cong End(R \oplus R)$ by (Nicholson & Yousif, 2003, Theorem 7.40). In particular $End(R \oplus R)$ is a right C2-ring, and this implies that $R \oplus R$ is C2 by (Nicholson & Yousif, 2003, Theorem 7.15).

$(2) \Rightarrow (1)$ is a consequence of Proposition 2.3 and (Nicholson & Yousif, 2003, Theorem 7.15).

A ring R is called a *left P-injective ring* if every homomorphism from a principal left ideal Rt to R can be extended to one from $_RR$ to $_RR$. Recall that a ring R is called a *left C2-ring* if every left ideal that is isomorphic to a direct summand of $_RR$ is also a direct summand of $_RR$. Every left P-injective ring is a left C2-ring (Nicholson & Yousif, 2003, Proposition 5.10). Recall from (Chen & Li, 2004, Theorem 2.21) that if R is left P-injective, left CIS and right noetherian, then R is QF. The next theorem generalizes (Chen & Li, 2004, Theorem 2.21).

Theorem 2.18. *If R is left P-injective, left CIS and right noetherian then R is QF.*

Proof. Clear by Proposition 2.3 because every left P-injective ring is left C2.

Nicholson and Yousif proved in (Nicholson & Yousif, 1998, Theorem 3.2) that R is QF if and only if R is a right Johns, left CS-ring. The next theorem generalizes (Nicholson & Yousif, 1998, Theorem 3.2), (Nicholson & Yousif, 1998, Theorem 3.4), (Chen et al., 2006, Theorem 10) and (Chen et al., 2006, Corollary 4).

Theorem 2.19. *The following are equivalent for a ring R:*

1. *R is QF;*
2. *R is a right Johns and a left CIS-ring;*
3. *R is left P-injective, left CIS-ring and right Noetherian ring;*
4. *R is left P-injective, left CIS-ring with ACC on left annihilators;*
5. *R is left P-injective, left CIS ring and satisfy ACC on right annihilators;*
6. *R is a right CIS-ring and every 2-generated right R-module embeds in a free module;*
7. *R is a left GP-injective, left CIS-ring with ACC on left annihilators.*

Proof. $(1) \Rightarrow (2)$ Since R is QF, then R is a right Johns and a left CIS-ring by (Nicholson & Yousif, 1998, Theorem 3.2).

$(2) \Rightarrow (3)$ Clear because every right Johns ring is right noetherian left P-injective.

$(3) \Rightarrow (1)$ R is QF by Theorem 2.18.

$(1) \Rightarrow (4)$ Clear by (Chen et al., 2006, Corollary 4).

$(4) \Rightarrow (3)$ Since R is a left P-injective ring (so is GP-injective) with ACC on left annihilators, then R is right Artinian by (Chen & Ding, 1999, Theorem 3.7), and hence it is right Noetherian.

$(1) \Rightarrow (5)$ Clear by (Chen et al., 2006, Theorem 10).

$(5) \Rightarrow (1)$ Since R is left CIS and left P-injective, then R is left CS. So, proof is clear by (Chen et al., 2006, Theorem 10).

$(1) \Rightarrow (6)$ Clear by (Nicholson & Yousif, 1998, Theorem 3.4).

$(6) \Rightarrow (1)$ By Proposition 2.8, R is a right artinian ring, and hence semiperfect with essential left socle. Then R is QF by (Nicholson & Yousif, 1998, Theorem 3.4(5)).

$(4) \Rightarrow (7)$ Clear because every left P-injective ring is left GP-injective.

$(7) \Rightarrow (1)$ Since R is a left GP-injective ring with ACC on left annihilators, then R is right Artinian by (Chen & Ding, 1999, Theorem 3.7), and hence it is right Noetherian. Now we want to show that every complement left ideal is left annihilator. Let C be a complement left ideal of R. Since R is left CIS, then there exist some $e^2 = e \in R$ such that $eR \cong C$. Since eR is left annihilator and $eR \cong C$, then C is a left annihilator. By (Chen et al., 2006, Theorem 3), R is QF.

The next theorem extends (Nicholson & Yousif, 2003, Theorem 8.9).

Theorem 2.20. *The following conditions are equivalent:*

1. *R is a right Johns, left Kasch ring;*
2. *R is a right CIS-ring and a right CF ring.*

Proof. $(1) \Rightarrow (2)$ Clear by (Nicholson & Yousif, 2003, Theorem 8.9).

$(2) \Rightarrow (1)$ R is right finitely cogenerated by (Nicholson & Yousif, 2003, Corollary 7.32) because it is right Kasch (being a right CF ring). Since R is right CF then every cyclic right R-module is finitely cogenerated. This implies that R is right artinian by Vámos' lemma (Nicholson & Yousif, 2003, Lemma 1.52), and so $Soc(_RR) \leq^e R_R$ (as R is semiprimary).

The following theorem generalizes (Yousif, 1997, Theorem 1.16).

Theorem 2.21. *The following conditions are equivalent for a ring R:*

1. *R is a right CIS-ring which is left and right Kasch;*
2. *R is right Kasch and a right continuous ring;*
3. *R is a semiperfect right continuous ring with essential right socle.*

Proof. $(1) \Rightarrow (2)$ By (Yousif, 1997, Lemma 1.15), R is a right C2-ring. By Corollary 2.4, R is right continuous.

$(2) \Rightarrow (1)$ Clear by (Yousif, 1997, Theorem 1.16).

$(2) \Leftrightarrow (3)$ is proved in (Yousif, 1997, Theorem 1.16).

The following theorem generalizes (Yousif, 1997, Theorem 1.17). We should point out that we use same technic used in (Yousif, 1997, Theorem 1.17) to proof the following theorem.

Theorem 2.22. *The following conditions are equivalent for a ring R:*

1. *R is a left and right Kasch left and right CIS-ring;*
2. *R is a left and right CIS-ring, and the dual of every simple right R-module is simple;*
3. *R is semiperfect left and right continuous ring with $Soc(_RR) = Soc(R_R)$ essential as a left and as a right R-module in R.*

Proof. $(3) \Rightarrow (1)$ Clear by (Yousif, 1997, Theorem 1.17).

$(1) \Rightarrow (3)$ By (Yousif, 1997, Lemma 1.15), R is a right C2-ring. By Proposition 2.3, R is a right CIS-ring. The proof follows by (Yousif, 1997, Theorem 1.17).

$(3) \Rightarrow (2)$ Clear by (Yousif, 1997, Theorem 1.17).

$(2) \Rightarrow (3)$ It is known that if the dual of every simple right R-module is simple, then R is a right Kasch ring. By (Yousif, 1997, Lemma 1.15), R is a left C2-ring. Then, by Corollary 2.4, R is a left continuous ring and hence semiregular by (Utumi, 1965). Now, since R is right Kasch and a right CIS-ring, then R has a finitely generated essential right socle, in particular R has no infinite sets of orthogonal idempotents by Proposition 2.6. Thus, R is semiperfect, and $Soc(_RR) \subseteq Soc(R_R)$. If e is a local idempotent of R, then $(eR/eJ)^* \cong l(J).e = Soc(R_R).e$ is a simple left R-submodule of Re. Since R is a left continuous ring then R is left CS. Thus, we have $Soc(R_R).e \leq^e Re$, for every local idempotent e of R. It implies that $Soc(Re) = Soc(R_R).e$ is simple and essential in Re, for every local idempotent e of R. Therefore, $Soc(_RR) = Soc(R_R)$ is essential as a left as well as a right R-module in R. By (Nicholson & Yousif, 1997, Lemma 4.16), R is left Kasch, and by (Yousif, 1997, Lemma 1.15), R is right continuous.

Theorem 2.23. *Let M be a left R-module, where $R = ReR$ for some idempotent $e \in R$ and $S = eRe$. Then:*

1. *The left R-module M is a CIS-module if and only if the left S-module eM is a CIS-module.*
2. *R_R is CIS if and only if the eRe-module Re is CIS.*

Proof. It can easily be proved routinely with using (Tercan & Yucel, 2015, Lemma 2.76 and Proposition 2.77).

Theorem 2.24. *$M_n(R)$ is CIS if and only if the free right R-module R^n is CIS.*

Proof. The result follows immediately by Theorem 2.23 but note that $M_n(R) = M_n(R)eM_n(R)$, where e is the matrix unit with 1 in the $(1,1)$th position and zero elsewhere.

The next theorem generalizes (Yousif, 1997, Theorem 1.18).

Theorem 2.25. *The following conditions are equivalent for a ring R:*

1. *R has a perfect duality;*
2. *R is left and right Kasch and $R \oplus R$ is a left and right CIS-module;*
3. *The dual of every simple right R-module is simple and $R \oplus R$ is a left and right CIS-module;*
4. *The dual of every simple right H-module is simple and H is a right and left CIS-ring, where $H = M_2(R)$.*

Proof. $(1) \Rightarrow (2)$, $(1) \Rightarrow (3)$ and $(1) \Rightarrow (4)$ clear by (Yousif, 1997, Theorem 1.18).

$(2) \Rightarrow (1)$ By (Yousif, 1997, Lemma 1.15) and Corollary 2.4, R is left and right continuous, so R is semiregular with $J(R) = Z(_RR) = Z(R_R)$ by (Utumi, 1965). Then by (Yousif, 1997, Lemma 1.1), $R \oplus R$ is left and right continuous as an R-module. By (Mohamed & Muller, 1990, Proposition 2.10) R is left and right self-injective ring. By Osofky's well known result (Osofsky, 1996), R has a perfect duality.

(3) ⇒ (4) By Theorem 2.24, $H = M_2(R)$ is a left (right) CIS-ring if and only if $R \oplus R$ is a left (right) CIS-module. By Theorem 2.22 and (Yousif, 1997, Lemma 1.2), R is a left (right) Kasch and left (right) mininjective ring. By Morita invariance, H is left (right) Kasch, and by (Nicholson & Yousif, 1997, Proposition 1.4), H is left (right) mininjective. Therefore, the dual of every simple right (left) H-module is simple by (Nicholson & Yousif, 1997, Proposition 2.2).

(4) ⇒ (1) By Theorem 2.22, H is semiperfect left and right continuous ring with $Soc(_H H) = Soc(H_H)$ essential as a left and as a right H-module. By (Nicholson & Yousif, 1997, Lemma 3.17) and (Utumi, 1965, Corollary 7.5), R has a perfect duality.

Corollary 2.26. *Suppose R is a left Kasch ring and ΠR is an arbitrary direct product of at least two copies of R. Then the following conditions are equivalent:*

1. *ΠR is a right CIS-ring;*
2. *ΠR is injective as a right R-module.*

Proof. (1) ⇒ (2) Since $R \oplus R$ is a right CIS-module, it is right continuous by (Yousif, 1997, Lemmas 1.1 and 1.15), and so R is right self-injective. So, ΠR is injective as a right R-module.

(2) ⇒ (1) Obvious.

We called a ring R is a right (countably) Σ-CIS-$ring$ if every direct sum of arbitrary (countably) many copies of R is CIS as a right R-module.

Corollary 2.27. *A ring R is quasi-Frobenius if and only if R is right and left Kasch and a right countably Σ-CIS-ring.*

Proof. Necessity is clear. For the converse let R be left Kasch and $(R \oplus R)_R$ be a CIS-module. Then by (Yousif, 1997, Lemmas 1.1 and 1.15), R is a right self-injective ring. Thus R is a semiperfect ring by (Osofsky, 1996). By (Dung et al., 1994, Corollary 8.11), R is right countably Σ-injective. So, R is quasi-Frobenius by (Faith, 1966).

ACKNOWLEDGEMENT

The researchers wish to express our deep sense of gratitude to Industrial University of Ho Chi Minh City for the financial support offered to this research project according to the Scientific Research Contract No 59/HĐ-ĐHCN, code 21.2CB01.

REFERENCES

Anderson, F. W. and Fuller, K. R. (1974). *Rings and Categories of Modules*. New York: Springer-Verlag.

Björk, J. E. (1969). *Rings satisfying a minimum condition on principal ideals*. J. Reine Angew. Math., 236, pp. 112-119.

Björk, J. E. (1972). *Radical properties of perfect modules*. J. Reine Angew. Math., 253, pp. 78-86.

Behboodi, M., Daneshvar, A. and Vedadi, M. R. (2018). *Virtually semisimple modules and a generalization of the Wedderburn-Artin theorem*. Comm. Algebra, 46 (6), pp. 2384-2395.

Chen, J., Shen, L. and Zhou, Y. (2006). *Characterizations of QF Rings*, Communications in Algebra, 35:1, pp. 281-288, DOI: 10.1080/00927870601042134.

Chen, J. and Ding, N. (1999). *On general principally injective rings*. Comm. Algebra 27, pp. 2097-2116.

Chen, J. L. and Li, W. X. (2004). *On artiness of right CF rings*. Comm. Algebra, 32(11), pp. 4485-4494.

Dung, N. V., Huynh, D. V., Smith, P. F. (1994). *Extending Modules.* Pitman RN Mathematics, Vol. 313, Harlow: Longman.

Faith, C. (1966). *Rings with ascending chain condition on annihilators*. Nagoya Math. J. 27, pp. 179-191.

Faith, C. (1976). *Algebra II, Ring theory.* New York: Springer-Verlag.

Jain, S. K., Srivastava, A. K. and Tuganbaev, A. A. (2012). *Cyclic modules and the structure of rings*. Oxford University Press.

Mohamed, S. H. and Muller, B. J. (1990). *Continuous and Discrete Modules*. London Math. Soc. Lecture Note Ser. No. 147, Cambridge Univ. Press, Cambridge, UK.

Nicholson, W. K., and Yousif, M. F. (1997). *Mininjective rings*. J. Algebra 187, pp. 548-578.

Nicholson, W. K., and Yousif, M. F. (1998). *Annihilators and the CS-condition*. Glasgow Mathematical Journal 40.2, pp. 213-222.

Nicholson, W. K., and Yousif, M. F. (2003). *Quasi-Frobenius Rings*. Cambridge Univ. Press.

Osofsky, B. L. (1996). *A generalization of quasi-Frobenius rings*. J. Algebra 4, pp. 373-387.

Padro, J. L. G., and Asensio, P. A. G. (1997). *Rings with finite essential socle*. Proc. Amer. Math. Soc. 125, pp. 971-

977.
Smith, P. F. (1992). *Modules for which every submodule has unique closure*. Proceedings of the Biennial Ohio-Denison Conference, pp. 302-313.
Tercan, A., and Yucel, C. C. (2015). *Module Theory, Extending Modules and Generalizations*. Birkhauser, Basel, Frontiers in Mathematics: Monograph Series.
Tolskaya, T. S. (1970). *When are all cyclic modules essentially embedded in free modules*. Mat. Issled., 5, pp. 187-192.
Utumi, Y. (1965). *On continuous rings and self-injective rings*. Trans. Amer. Math. Soc. 118 , pp. 158-173.
Yousif, M. F. (1997). *On continuous rings*. J. Algebra 191, pp. 495-509.

VÀNH VÀ MÔĐUN CÓ PHẦN BÙ ĐẲNG CẤU VỚI HẠNG TỬ TRỰC TIẾP CỦA CHÍNH NÓ

NGUYỄN THỊ THU HÀ
Khoa Khoa học Cơ bản, Trường Đại học Công nghiệp Thành phố Hồ Chí Minh.
nguyenthithuha@iuh.edu.vn

Tóm tắt. Trong bài báo này chúng tôi giới thiệu một mở rộng của điều kiện CS nổi tiếng. Chúng tôi gọi một *R*-môđun *M* là *CIS-môđun* nếu mọi phần bù của nó đều đẳng cấu với một hạng tử trực tiếp. Chúng tôi chứng minh được rằng nếu *R* là một vành FGF phải CIS phải, thì *R* là vành tựa Frobenius, và nếu *R* là vành CF phải CIS phải thì *R* là vành Arin phải. Tính chất mới của vành tựa Frobenius được đưa ra bằng cách sử dụng các vành CIS. Hơn nữa, nhiều mệnh đề quan trọng liên quan đến vành CS được mở rộng thành vành CIS cũng được chúng tôi trình bày.
Từ khóa. CIS-môđun, vành CIS, vành tựa Frobenius, vành CF, vành FGF.

Received on: 26/04/2022

Accepted on: 01/07/2022